Randomized Algorithms – Problem Set 3

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Homework. Please keep the answers to the following questions short and easy to read.

1. Prove that in any graph $G = (V, E)$ there is an independent set of size at least $\sum_{v \in V} \frac{1}{degree(v)+1}$

2. Consider the simultaneous message model for evaluating a function $f(x, y)$: Alice and Bob share a random string. They receive inputs x and y respectively and each should send a message to a referee, Charlie, who should evaluate the function $f(x, y)$. They may also have their own private source of randomness. The goal is for Alice and Bob to send short messages to Charlie.

   We will consider the equality function, i.e. $x, y \in \{0,1\}^n$ and $f(x, y) = 1$ if $x = y$ and 0 otherwise.

   (a) Recall the protocol seen in class where Alice and Bob send to Charlie an inner product of their input with a common random string r. What happens to this protocol if Eve, who selects the inputs and whose goal is to make Charlie compute the wrong value, knows the common string r when she selects x and y?

   (b) Suggest a non-trivial protocol for this case, where Eve knows the common random string but not the private source of randomness that Alice and Bob each have. By non-trivial we mean one with message length which is sublinear in the input length and probability of Charlie being correct at least 2/3 for any pair of inputs chosen by Eve.

   Hint: you may use the fact that there are good error correcting codes $C : \{0,1\}^n \mapsto \{0,1\}^m$ where m is $O(n)$ and for any two different strings $x_1, x_2 \in \{0,1\}^n$ the distance between $C(x_1)$ and $C(x_2)$ is $\Omega(n)$.

3. Consider the following family of functions H where each member $h \in H$ is such that $h : \{0,1\}^\ell \mapsto \{0,1\}$. The members of H are indexed with a vector $r \in \{0,1\}^{\ell+1}$. The value $h_r(x)$ for $x \in \{0,1\}^\ell$ is defined by considering the vector $x' \in \{0,1\}^{\ell+1}$ obtained by appending 1 to x and the value is $\langle r, x' \rangle$ - the inner product of r and x' over $GF[2]$.

   Prove that the family H is three-wise independent.

4. Prove that for *any* hash function $h : \{0,1\}^* \mapsto \{0,1\}^\ell$, the expected time (i.e. evaluations of h) to find a collision is $O(2^{\ell/2})$.

5. A hundred people are in line to enter a movie theater with a hundred seats. Each person holds a ticket with an assigned seat. The first in line drops his ticket and, instead of looking for it, sits in a random place. The others enter the theater one by one and each one, if their seat is taken, instead of arguing, sits in a random vacant seat. What is the probability that the last person in line will sit in her assigned seat?