Algorithmic Game Theory – Handout 12

Uriel Feige, Robert Krauthgamer, Moni Naor

January 28, 2009

We study cooperation games, where the goal is to sustain cooperation among agents. The main question is how to split the benefits or cost of a joint project among the agents. Today's focus will be cost-sharing games, such as when agents in a network are interested in receiving a certain service – the total cost of the service depends on which agents actually receive the service, it should be divided among those receiving the service. We consider solution concepts that preclude "defection" by any subset of the agents, and mechanisms that elicit truthful bids from the agents.

Reading. More information on cost sharing can be found in [NRTV, Chapter 15].

Homework. Answer the following questions. You are advised to read [IMM], especially the parts relevant to the questions below. Please keep the answers short and easy to read.

The following cost-sharing game is called *the set cover game*: There is a set A of n agents, and a collection $\mathcal{E}$ of subsets of A (that together covers all of A i.e. $\cup_{T\in\mathcal{E}} T = A$). For every subset of agents $S \subseteq A$, the cost $c(S)$ is minimum number of sets from $\mathcal{E}$ that covers all of S, namely the minimum k such that there are k sets $T_1, \dots, T_k \in \mathcal{E}$ with $S \subseteq \cup_{l=1}^{k} T_l$.

1. The following is a standard integer program for $c(A)$:

$$\begin{aligned}
&\max \sum_{T\in\mathcal{E}} x_T && \\
&\text{s.t.} \sum_{T\in\mathcal{E}: j\in T} x_T \geq 1 && \forall j \in A \\
&x_T \in \{0,1\} && \forall T \in \mathcal{E}
\end{aligned}$$

   Show that relaxing the last constraint to $x_T \geq 0$ gives a linear program whose integrality gap is larger than any constant (when n is sufficiently large), i.e. there are instances where the fractional and integral solutions to this LP differ by a factor that is increasing with n. How does it grow with n?

   Hint: Consider $A = \{1, \dots, n\}$ and for each $i = 1, 2, \dots$ let $T_i \subseteq A$ contain all $j \in A$ whose ith bit (in the binary representation of j) equals 1.

2. What is the smallest $\gamma = \gamma(n) > 0$ for which there are instances of the set cover game with an empty γ-core?

   Hint: Using the previous question, show equivalence between the appropriate LPs.

3. Consider the special case of the set cover game where every agent $j \in A$ belongs to (can be covered by) at most two sets in $\mathcal{E}$. Show that this game admits a group strategyproof mechanism that is γ-budget balanced for $\gamma \geq \Omega(1/\sqrt{n})$ (or at the very least, $\gamma \gg 1/n$).

Hint: Analyze the following cost-sharing scheme for cross-monotonicity and γ-budget-balance:

$$\xi(j,S) = \min_{T:j\in T} \frac{1}{|T \cap S|}.$$

4. Prove or disprove: The set cover game is a submodular game (i.e. for all instances).

   Recall: a cost-sharing game (A, c) is called submodular if the cost function c satisfies

   $$\forall S, T \subseteq A, \quad c(S) + c(T) \geq c(S \cup T) + c(S \cap T).$$

   This condition is equivalent to having a decreasing marginal cost.

References

[NRTV] Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay V. Vazirani (Editors), Algorithmic Game Theory, Cambridge University Press, 2007.

[IMM] Nicole Immorlica, Mohammad Mahdian, and Vahab S. Mirrokni, Limitations of cross-monotonic cost-sharing schemes. ACM Transactions on Algorithms 4(2), 2008.