Bilevel Nonlinear Programs

by Michael Kupferschmid

This paper investigates three approaches for solving optimization problems of this form.

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y})$$
$$\text{subject to} \begin{cases} f_i(\mathbf{x},\mathbf{y}) \leq 0,\ i = 1\ldots m_O \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) \\ \text{subject to } g_i(\mathbf{y};\mathbf{x}) \leq 0,\ i = 1\ldots m_I \end{array} \right] \end{cases}$$

1 The Graphical Approach

Consider the optimization problem given below (it is Test Problem 13 of §7). The variable y is constrained to be an optimal point for the **inner problem**, which is enclosed in square brackets. The **outer problem** or overall optimization is solved by varying both x and y, but in the inner problem the variable x is treated as a constant parameter and the optimization is performed by varying only y.

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x - \tfrac{13}{4})^2 + (y-2)^2$$
$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + \frac{3}{2} \leq 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - \frac{45}{8} \leq 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-8)^2 + \frac{1}{2}xy^2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -3x + y + 3 \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = \frac{5}{3}x - y - 8 \leq 0 \\ g_3(\mathbf{y};\mathbf{x}) = x + y - 7 \leq 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y \leq 0 \end{cases} \end{array} \right] \end{cases}$$

At each $x \in [1, \frac{45}{8}]$ the inner problem of this example is feasible and not unbounded, so for those values of x it has a minimizing point $y^\star(x)$.

This example is simple enough that for a given value of x the inner minimization can be performed by searching the line

$$L(x) = \{y \mid g_i(y;x) \le 0,\ i = 1 \ldots 4\}$$

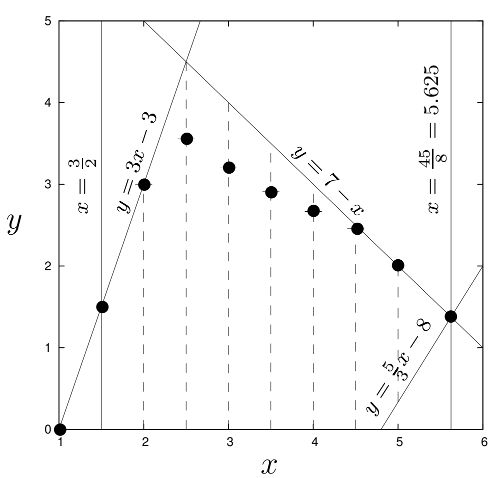
for the point $y^\star(x)$ where $g_0(y;x)$ attains its lowest value. That will be where

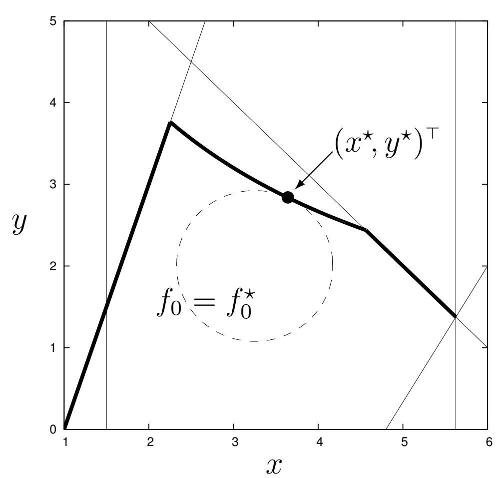
$$\begin{aligned} \frac{dg_0}{dy} = 2(y-8) + xy &= 0 \\ \text{or} \qquad y &= \frac{16}{2+x}, \qquad x \ne -2, \end{aligned}$$

if that point is interior to $L(x)$, or at an endpoint of $L(x)$ otherwise. Calculations to find $y^\star(x)$ are displayed for several values of x in this table.

x	$L(x)$ bottom		unconstrained		$L(x)$ top		$y^\star(x)$
	y	$g_0(y;x)$	$y = 16/(2+x)$	$g_0(y;x)$	y	$g_0(y;x)$	
1.00	0.00	64.0	5.33	$(x,y)^\top$ infeasible	0.00	64.0	0.00
1.50	0.00	64.0	4.57	$(x,y)^\top$ infeasible	1.50	43.9	1.50
2.00	0.00	64.0	4.00	$(x,y)^\top$ infeasible	3.00	34.0	3.00
2.50	0.00	64.0	3.56	35.6	4.50	37.6	3.56
3.00	0.00	64.0	3.20	38.4	4.00	40.0	3.20
3.50	0.00	64.0	2.91	40.7	3.50	41.7	2.91
4.00	0.00	64.0	2.67	42.7	3.00	43.0	2.67
4.50	0.00	64.0	2.46	44.3	2.50	44.3	2.46
5.00	0.33	59.1	2.29	$(x,y)^\top$ infeasible	2.00	46.0	2.00
5.63	1.38	49.2	2.10	$(x,y)^\top$ infeasible	1.38	49.2	1.38

The left figure below shows the constraint hyperplanes of the problem, along with dashed line segments representing $L(x)$ for the tabled values of x and dots (•) at the coordinates $(x, y^\star(x))$ given in the table. The line $L(1)$ is the point $(1,0)^\top$; $L(\frac{45}{8})$ is the point $(\frac{45}{8}, \frac{11}{8})^\top$.





Finding $y^\star(x)$ for additional values of x leads to the picture on the right. The dark edges are the **inducible region** $y^\star(x)$ for the inner problem, so named because it is induced by the inner optimization and thus not necessarily coincident with any constraint contour (although this inducible region happens to overlap parts of two constraint contours). Once the inner problem's inducible region is known, the overall problem of this example can be solved graphically. The dot in the right picture is the optimal point $(x^*, y^*)^\top \approx (3.66, 2.83)^\top$ and the dashed circle is the optimal contour $f_0 = f_0^\star \approx 0.852$ of the outer objective.

This problem is a [9] cooperative or optimistic Stackelberg game or [19, §1.6] **bilevel program**. In general a bilevel program can have n_x variables x_j, n_y variables y_j, m_O outer constraints, and m_I inner constraints. The constraints can include equalities as well as inequalities, and the f_i and g_i need not be linear. The inducible region for this problem happens to be connected but, as shown by some of the examples in §7, an inducible region need not be connected. The optimal point for this problem happens to be interior to the feasible set but, as shown by some of the examples in §7, an optimal point can be in the boundary. The graphical solution technique is useful only for the simplest of problems.

Bilevel programs arise in many applications, and effective (though computationally demanding) methods have been devised for solving those in which all of the functions are linear [16] [9]. Algorithms have also been proposed for solving bilevel programs in which some of the functions are nonlinear [14], but the development of methods for those much more difficult problems remains an active area of research.

2 The MPEC Approach

Some bilevel programs can be solved by replacing the constraint that $\mathbf{y}$ solves the inner problem by the constraint that $\mathbf{y}$ solves the Karush-Kuhn-Tucker conditions for the inner problem. For our example the KKT reformulation looks like this.

$$
\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x - \tfrac{13}{4})^2 + (y-2)^2
$$

$$
\text{subject to} \left\{
\begin{array}{lcl}
f_1(\mathbf{x},\mathbf{y}) & = & -x + \frac{3}{2} \le 0 \\
f_2(\mathbf{x},\mathbf{y}) & = & x - \frac{45}{8} \le 0 \\
g_1(\mathbf{y};\mathbf{x}) & = & -3x + y + 3 \le 0 \\
g_2(\mathbf{y};\mathbf{x}) & = & \frac{5}{3}x - y - 8 \le 0 \\
g_3(\mathbf{y};\mathbf{x}) & = & x + y - 7 \le 0 \\
g_4(\mathbf{y};\mathbf{x}) & = & -y \le 0 \\
h_0(\mathbf{y};\mathbf{x}) & = & 2(y-8) + xy + u_1 - u_2 + u_3 - u_4 = 0 \\
h_1(\mathbf{y},\mathbf{u};\mathbf{x}) & = & u_1(-3x + y + 3) = 0 \\
h_2(\mathbf{y},\mathbf{u};\mathbf{x}) & = & u_2(\frac{5}{3}x - y - 8) = 0 \\
h_3(\mathbf{y},\mathbf{u};\mathbf{x}) & = & u_3(x + y - 7) = 0 \\
h_4(\mathbf{y},\mathbf{u};\mathbf{x}) & = & u_4(-y) = 0 \\
u_i \ge 0, & & i = 1, \ldots, 4
\end{array}
\right.
$$

The KKT system for the inner problem includes a stationarity condition *only* with respect to y,

$$h_0 = \frac{\partial}{\partial y}\left(g_0(y;x) + \sum_{i=1}^{4} u_i g_i(y;x)\right) = 0$$

because x is a constant parameter, not a variable, in the inner optimization. The orthogonality conditions $h_i = 0$ for $i \geq 1$ and the nonnegativities $u_i \geq 0$, taken together, are often referred to as **complementarity constraints**, and because of them the KKT reformulation is referred to as a mathematical program with equilibrium constraints or **MPEC**.

It is easy to verify that the optimal point for our example (which is given in §7) satisfies the constraints of this one-level nonlinear program, but it is *not* easy to solve the KKT reformulation for $(x^\star, y^\star, \mathbf{u}^\star)$. At the optimal point the equalities $h_i = 0$ and the nonnegativities $u_i \geq 0$ are all tight, but their gradients are *not* linearly independent. In fact, the KKT reformulation of a bilevel nonlinear program might [13] [26] not satisfy *any* constraint qualification [22] so the KKT theory cannot in general be used a second time to solve the one-level problem analytically. Because the orthogonality conditions are nonlinear equalities, the KKT reformulation is also quite challenging for general-purpose numerical optimization algorithms [3] [14, §2.2].

Of course the KKT reformulation cannot be used at all unless the inner problem has a constraint qualification at the optimal point of the bilevel program (our example satisfies Slater's condition). If the inner problem is nonconvex there might be more than one $\mathbf{y}$ that satisfies its KKT conditions at $\mathbf{x}^\star$, and some might not be even local minima (our inner problem has a nonconvex objective, but only one point satisfies its KKT conditions and it is a minimum). If at the optimal point for the bilevel program more inner constraints are active than there are inner variables, then the KKT multipliers $\mathbf{u}$ will not be uniquely determined. Despite these difficulties, powerful algorithms have been developed [20] [14] for solving MPECs numerically, including MPECs resulting from the KKT reformulation of linear bilevel programs.

3 The Substitution Approach

If we could solve the inner problem explicitly for $\mathbf{y}^\star(\mathbf{x})$, that formula could be substituted for $\mathbf{y}$ in the outer problem to yield a one-level optimization involving only $\mathbf{x}$. In our example, if we knew ahead of time that the optimal point would turn out to be in the part of the inducible region where $y = 16/(2+x)$, we could substitute that formula in the bilevel program to obtain this one-level reformulation.

$$\min_{\mathbf{x}} f_0(\mathbf{x}) = (x - \tfrac{13}{4})^2 + \left(\frac{16}{2+x} - 2\right)^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x}) = -x + \tfrac{3}{2} \leq 0 \\ f_2(\mathbf{x}) = \quad x - \tfrac{45}{8} \leq 0 \end{cases}$$

Usually the inducible region is not known analytically, but for each trial point $\mathbf{x}^k$ that is proposed by some numerical solver of the outer problem, a numerical solver of the inner problem might compute the corresponding $\mathbf{y}^k = \mathbf{y}(\mathbf{x}^\mathbf{k})$ in this reformulation.

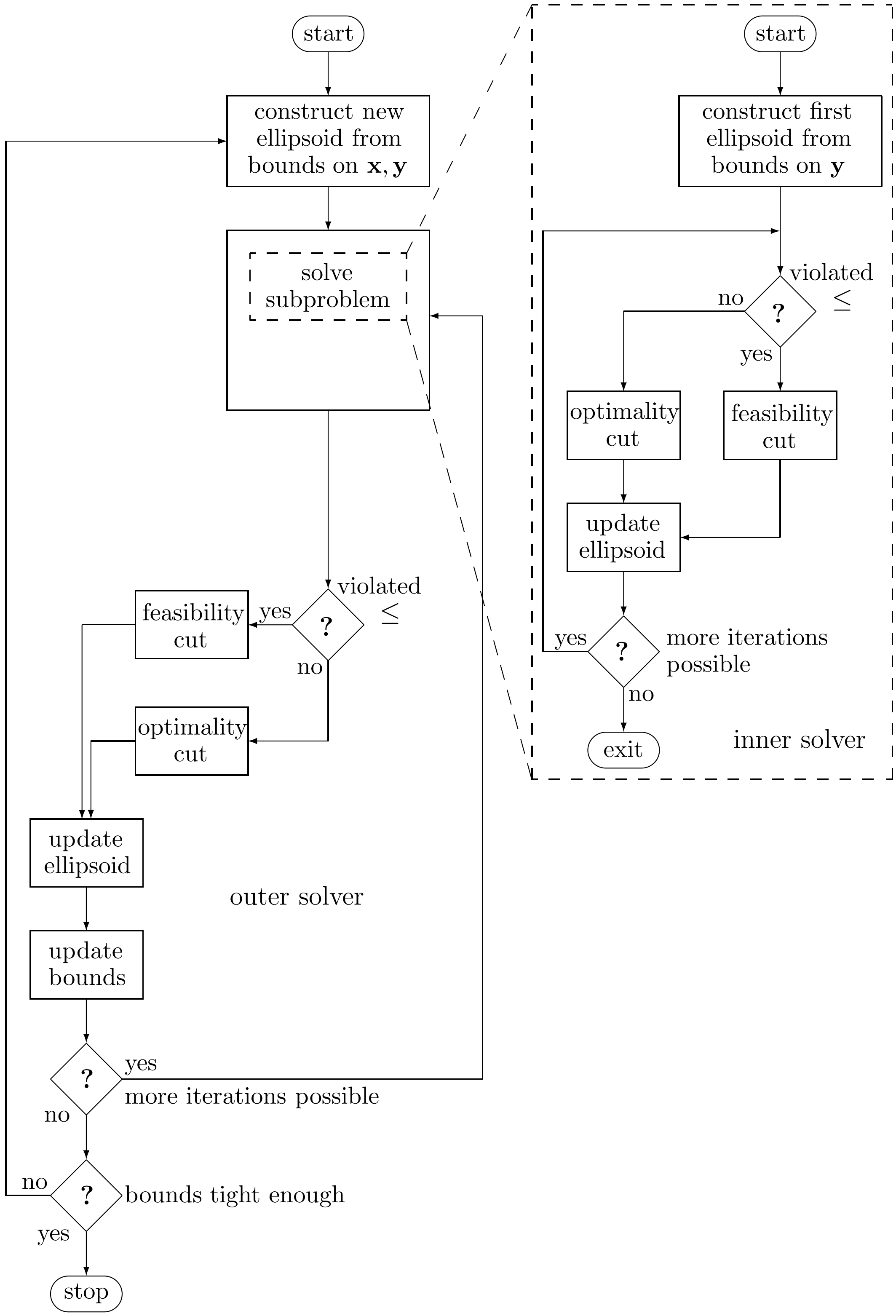
$$\min_{\mathbf{x}} f_0(\mathbf{x}, \mathbf{y}(\mathbf{x}))$$
$$\text{subject to} \begin{cases} f_i(\mathbf{x}, \mathbf{y}(\mathbf{x})) \leq 0, \; i = 1 \ldots m_O \\ g_i(\mathbf{x}, \mathbf{y}(\mathbf{x})) \leq 0, \; i = 1 \ldots m_I \\ \mathbf{y}(\mathbf{x}) \text{ solves} \left[\begin{array}{l} \min_{\bar{\mathbf{y}}} g_0(\bar{\mathbf{y}}; \mathbf{x}) \\ \text{subject to } g_i(\bar{\mathbf{y}}; \mathbf{x}) \leq 0, \; i = 1 \ldots m_I \end{array} \right] \end{cases}$$

A practical implementation of this approach must do something sensible if the outer solver happens to propose an $\mathbf{x}^k$ at which the inner problem is infeasible or unbounded, or at which the inner problem has multiple optima. It also requires that we finite-difference function values to compute derivatives of $\mathbf{y}(\mathbf{x})$ for the solver of the outer problem.

4 Computational Experiments

To investigate the utility of the substitution approach I wrote a computer program, referred to here as `bnlezy`, that uses the ellipsoid algorithm [19, §24] to solve both the inner problem and the outer problem. This choice was motivated by the observation in [11] that the algorithm often finds an optimal point even of a nonconvex problem. A block diagram of the implementation is shown on the next page.

The ellipsoid algorithm begins by constructing an ellipsoid passing through the corners of a box defined by upper and lower bounds on the variables. If the center of the ellipsoid violates an inequality constraint, the algorithm constructs a hyperplane supporting the constraint contour at that point. If the center of the ellipsoid is feasible, the algorithm constructs a hyperplane supporting the objective contour at that point. In either case the ellipsoid is cut in half by the hyperplane, and a next ellipsoid is constructed as the smallest one containing the half of the old ellipsoid that is on the feasible or optimal side of the cutting hyperplane. The cutting hyperplane can also be used to update the bounds on the variables, defining a region of uncertainty within which, if the problem is convex, the minimizing point must lie. This process can be continued, producing smaller ellipsoids and tighter bounds, until the largest uncertainty in the optimal value of a variable is small enough. If the matrix defining the ellipsoid becomes numerically non-positive-definite before then, it is sometimes possible to restart the algorithm by constructing a new ellipsoid enclosing the current variable bounds. The algorithm keeps track of the feasible ellipsoid center having the lowest objective value found so far, and this is referred to as the **record point** [19, §9.6].


start
construct new ellipsoid from bounds on $\mathbf{x}, \mathbf{y}$
solve subproblem
violated ≤
?
yes
no
feasibility cut
optimality cut
update ellipsoid
update bounds
outer solver
?
yes
more iterations possible
no
?
no
bounds tight enough
yes
stop
start
construct first ellipsoid from bounds on $\mathbf{y}$
violated ≤
no
?
yes
optimality cut
feasibility cut
update ellipsoid
yes
?
more iterations possible
no
exit
inner solver

When an inner problem is infeasible I use the final ellipsoid center found by the algorithm as its solution, on the assumption that this point is the one that is closest to being feasible. When an inner problem has multiple optima I use the one found by the ellipsoid algorithm, on the assumption that it is the point (or a point) having the lowest objective value. If these assumptions are unfounded, or if an inner problem is unbounded, the algorithm fails.

The test suite for my experiments consists of the 30 problems described in §7, many of which are deviously contrived to exhibit specific pathologies. I ran `bnlezy` on each problem, starting from both **fair bounds** and **tight bounds** on the variables. The fair bounds are deduced from the constraints of the problem or determined using the rules described in [19, §26.2.2] and the tight bounds are very tight around the known optimal point.

In each experiment where a record point $(\mathbf{x}^r, \mathbf{y}^r)^\top$ was found by the outer solver I calculated the amount by which it differs from the known optimal point $(\mathbf{x}^\star, \mathbf{y}^\star)^\top$ according to this formula.

$$\delta^r = \sum_{j=1}^{n_x} (x_j^r - x_j^\star)^2 + \sum_{j=1}^{n_y} (y_j^r - y_j^\star)^2$$

The bounds on the variables place the starting point much farther from the optimal point in some problems than in others, so I also found the error δ° at the starting point of each problem and used it to calculate the log relative error of a record point from this formula.

$$\Delta^r = \log_{10}(\delta^r/\delta^\circ)$$

This error measure is zero at the starting point and has negative values at points closer to the optimal point. If the record point $(\mathbf{x}^r, \mathbf{y}^r)^\top$ is precisely (in floating-point numbers) the optimal point $(\mathbf{x}^\star, \mathbf{y}^\star)$, then $\Delta^r = -\texttt{Inf}$. If the starting point of a problem is infeasible, the ellipsoid algorithm might generate iterates farther away from the optimal point than the starting point was, and if $(\mathbf{x}^r, \mathbf{y}^r)$ is one of those iterates then $\Delta^r > 0$. When a test problem has alternate optima, the $(\mathbf{x}^\star, \mathbf{y}^\star)$ that I used in computing the solution error was the optimal point closest to the $(\mathbf{x}^r, \mathbf{y}^r)$ reported by the algorithm.

The table on the next page reports Δ^r for each experiment, or a blank if no record point was found. The MPEC reformulation of each problem in this test suite is solved in [18] by the production codes MINOS 5.5 and filterSQP, neither making use of bounds on the variables, and I have included the log relative errors of those solutions in the table for comparison. If we consider an algorithm to be successful when it reduces the starting distance from the optimal point by a factor of 1000 or more, so that $\Delta^r \leq -3$, then we get the bottom row of the table. By this measure `bnlezy`, MINOS 5.5, and filterSQP are each successful on only about 2/3 of the problems in this set.

problem number	`bnlezy` fair	`bnlezy` tight	`MPEC` MINOS	`MPEC` filterSQP
1	-24.25	-24.11	-Inf	-Inf
2	-25.69	-24.64	-Inf	-Inf
3	-15.38	-11.26	-Inf	-Inf
4	-16.97	-13.81	-0.84	-11.15
5	-10.33	-4.61	-Inf	-Inf
6	0.69	1.93	-Inf	-Inf
7	-10.45	-7.07	-Inf	-Inf
8	0.00	-12.69	-Inf	-Inf
9	-0.92	1.92	-20.92	-32.21
10	-14.45	3.91	-Inf	-Inf
11	-31.68	-11.54	-Inf	-Inf
12	-11.40	-19.87	-Inf	-Inf
13	-7.40	-3.88	-10.40	-10.40
14	-26.28	-22.42	-7.19	-7.19
15	-24.02	-20.76	-11.31	-11.31
16	-11.25	-6.88	1.09	-Inf
17	-18.61	-11.12	-14.06	-14.06
18	-13.94	-8.43	-7.86	-1.05
19	-16.68	2.02	-0.01	-0.01
20	-10.91	2.01	-0.08	-0.08
21	-23.16	-15.19	-12.75	-12.75
22	0.60	1.92	-15.40	-Inf
23	-10.41	-6.57	-Inf	-Inf
24	-0.58	1.64	-Inf	-Inf
25	-26.91	-20.06	-1.83	-1.83
26	-11.44	-4.02	-12.64	-12.64
27	-16.18	-11.86	-16.40	-30.20
28	-6.21	-3.41	-0.12	-0.12
29	0.10	-23.87	0.20	0.33
30	-0.12		1.74	1.74
% solved	77	73	73	77

5 Discussion

Starting from fair bounds, `bnlezy` fails on seven of the problems.

- Problem 6 has its starting point at $x = 2.75$, for which the inner solver returns $y^\star(x) \approx 0$ but positive; to minimize the outer objective the algorithm then increases x to its upper bound. If the starting point is instead to the left of the vertical segment in the inducible region, `bnlezy` converges to the optimal point.

- Problem 8 has an infimum at the origin yielding the optimal objective value, and `bnlezy` converges to that point even though it is not in the inducible region.

- Problems 9 and 22 have inner problems with multiple optimal points $y^\star(x)$ when $x = x^\star$, corresponding to vertical segments of their inducible regions. In Problem 9 the inner solver returns $y^\star(0) = 1$ rather than $y^\star(0) = 0$, and in Problem 22 the inner solver returns $y^\star(0) = -1$ rather than $y^\star(0) = 0$; these alternate optimal solutions of the inner problems make it impossible for `bnlezy` to solve the overall optimizations.

- Problem 24 yields $x = x^\star = 5$, $y_1 = y_1^\star = 4$, $y_2 = 3.3$, with an inner objective value of $g_0(x^\star, y^\star) = -4$, but at $x = 5$ the inner problem also has an objective value of $g_0(x^\star, y^\star) = -4$ at $y_2 = y_2^\star = 2$, so once again the difficulty is that the inner problem has multiple optimal solutions at $x^\star$.

- Problem 29 yields $x_1 = 25$, $x_2 = 30$, $y_1 = 5$, $y_2 = 10$ with an outer objective value of $f_0(x^\star, y^\star) = +5$, and problem 30 yields $x_1 = 0.5$, $x_2 = 0.4$, $y_1 = y_2 = y_3 = 0$ with an outer objective value of $f_0(x^\star, y^\star) = -5.6$, but these problems are large enough that it is hard to diagnose why `bnlezy` fails.

The substitution approach involves the approximate optimization of nonlinear subproblems and the approximate calculation of gradients, here by central differencing. When the algorithm works at all, these numerical processes might limit the precision of the solutions it finds or affect its stability. To investigate these effects I solved all of the problems from tight bounds containing the optimal point, on the assumption that starting close to an answer makes it more likely to be found. Surprisingly, the summary statistics reported above suggest that for the problems in this test suite `bnlezy` is less good at refining a near-optimal starting point than it is at finding an approximate solution from farther away. However, because the solution errors Δ^r tabulated above are relative to the error δ^0 at the starting point of each problem, it is much more difficult to achieve $\Delta^r \leq -3$ for the tight bounds than for the fair bounds. Also, the intersection of each problem's feasible region with its tight bounds is much smaller than the intersection with its fair bounds, so the starting ellipsoids are smaller and [11] the ellipsoid algorithm likely does not sample as much of $\mathbb{R}^{n_x+n_y}$.

Most of the problems that `bnlezy` fails to solve for one or both sets of bounds are solved by either MINOS or filterSQP, and all but one of the problems that give one or the other of those algorithms trouble are solved by `bnlezy` for at least one set of bounds. This suggests that when the MPEC and substitution approaches fail it is for different reasons. It also appears that problem 30 has some property that makes it difficult for all three algorithms.

These results suggest several questions that might be addressed in a future study of the substitution approach.

- Would some method other than the ellipsoid algorithm be a better choice for solving the inner or outer problem?

- Plotting the iterates (x^k, y^k) generated by the algorithm on top of a two-dimensional problem's contour diagram might shed light on why the approach fails when $\mathbf{y}^\star$ is not unique. Does it suggest some modification that would ensure success?

- If an inner problem solution $\bar{\mathbf{y}}$ is infeasible but a record point is known, a backtracking linesearch [19, §19.1] might be used to find an $\hat{\mathbf{x}}$ for which the inner problem *is* feasible and $(\hat{\mathbf{x}}, \mathbf{y}^\star(\hat{\mathbf{x}}))^\top$ is a new record point. Does this improve the performance of the algorithm?

- How does the comparison with the MPEC approach change if the production solvers are provided with the same bounds on the variables that are used to determine a starting point for the ellipsoid algorithm?

6 Acknowledgements

This paper is a shortened version of one on which Tyge Rugenstein was co-author. That paper was never published, but John Mitchell and two anonymous referees read it and made helpful suggestions. Caroline Kim and Erik Fast performed the computational experiments [18] involving MINOS and filterSQP.

7 The Test Problems

This section contains a precise statement of each problem including its source, its known optimal point(s), and the initial variable bounds that I used. For each problem having $n_x + n_y = 2$ variables, a contour diagram is also provided illustrating the constraints (light lines), the fair starting point ($\times$), the inducible region (dark lines), the optimal point ($\bullet$), and the optimal objective function contour (dashed line). Each contour diagram is accompanied by an analytic description of the inducible region $y^\star(x)$.

I included (as Problems 1,5,11,12,14,23,24,25,26,29) all of the nonlinear bilevel programming test problems in [15, §9.3], except for their problem 4 (§9.3.5), which contains an equality constraint in its inner problem, and their problem 7 (§9.3.8), which is identical to their problem 1 (§9.3.2). To these I added all of the numerical examples in [22] (my Problems 8,10,18,30) and in [9] (my Problems 2,6,7,9,15,17,19,20,21,22,27), a problem from [4] (my Problem 28), and a few (my Problems 3,4,13,16) that were made up for this paper. I chose these problems because they are simple enough to be used for exploring the behavior of the algorithms in minute detail; it was not my intention in this paper to test the methods exhaustively on large problems or real applications. Many of these test problems are difficult despite their low dimensions, on account of the various interesting pathologies they exhibit.

Some test problems appear repeatedly in the literature under different names, often with small variations, so to make it clear how the earlier problem statements are related to each other and to mine I have carefully described the origin of each problem in this collection. The symbol $\leftarrow$ indicates that the problem was copied unchanged from the reference cited on the right of the arrow to the one cited on the left. The symbol $\leftsquigarrow$ indicates that the problem given in the reference cited on the left was derived from or is related to the problem given in the reference cited on the right but is not precisely the same.

Each problem statement includes starting bounds $\mathbf{x}^H$, $\mathbf{x}^L$, $\mathbf{y}^H$, and $\mathbf{y}^L$ on the variables. The **natural bounds** are deduced from the given constraints, but sometimes these cannot be used to determine a starting point and starting ellipsoid for `bnlezy`. One or more of the natural bounds might be $\pm\infty$, or their midpoint might place one or more of the variables at its optimal value, or they or their midpoint might not be exactly representable as floating-point numbers. I therefore compute the fair bounds based on the natural bounds using the set procedure [19, §[26.2.2] mentioned above. I made the tight bounds straddle the optimal values of the variables with small variations in the second significant digit, by adding the same random vector to both the upper and lower bounds; thus their midpoint is close to but not exactly at the optimal point.

The values $u_i^\star$ given in each problem statement are the KKT multipliers corresponding to the constraints g_i of the inner problem and satisfying the KKT conditions for the inner problem at the optimal point. I found them analytically by writing the KKT conditions for the inner problem (assuming that $\mathbf{x}$ is fixed), substituting the known $\mathbf{x}^\star$ and $\mathbf{y}^\star$, and solving for $\mathbf{u}^\star$. When the multipliers were not uniquely determined, I chose values in accordance with the advice given in [20].

Bounds are also given on the KKT multipliers, for possible use in an algorithm that solves the MPEC reformulation. In all cases I assumed natural bounds on the multipliers of $u_i^H = +\infty$ and $u_i^L = 0$, for which my set procedure yields fair bounds of $u_i^L = 0$ and

$$u_i^H = \begin{cases} 11 \times u_i^\star & u_i^\star > 0 \\ 1 & u_i^\star = 0. \end{cases}$$

The tight bounds I suggest for the KKT multipliers are always $u_i^L = 0$ and

$$u_i^H = \begin{cases} 2 \times u_i^\star & u_i^\star > 0 \\ 1 & u_i^\star = 0. \end{cases}$$

The multiplier values $u_i^\star$ satisfying the inner problem's KKT conditions at $(\mathbf{x}^\star, \mathbf{y}^\star)$ are used only for determining these bounds.

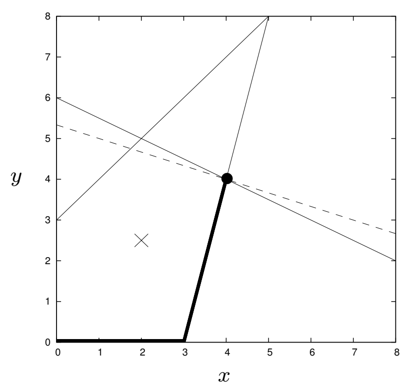
Test Problem 1

$$
\begin{aligned}
&\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -x - 3y \\
&\text{subject to} \left\{ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y \\ \text{subject to} \left\{ \begin{array}{l} g_1(\mathbf{y};\mathbf{x}) = -x + y - 3 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = x + 2y - 12 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 4x - y - 12 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -x \le 0 \\ g_5(\mathbf{y};\mathbf{x}) = -y \le 0 \end{array} \right. \end{array} \right] \right.
\end{aligned}
$$

Starting Bounds

	natural	fair	tight
x^H	4	4.0000000000000000E+00	4.1004915073699495E+00
x^L	0	0.0000000000000000E+00	3.9004915073699498E+00
y^H	5	5.0000000000000000E+00	4.1098691460679877E+00
y^L	0	0.0000000000000000E+00	3.9098691460679875E+00

Solution



$$
y^\star(x) = \begin{cases} 0 & 0 \le x \le 3 \\ 4x - 12 & 3 \le x \le 4 \end{cases}
$$

$$
\begin{aligned}
x^\star &= \texttt{4.0000000000000000E+00} \\
y^\star &= \texttt{4.0000000000000000E+00} \\
f_0^\star &= \texttt{-1.6000000000000000E+01} \\
u_1^\star &= \texttt{0.0000000000000000E+00} \\
u_2^\star &= \texttt{0.0000000000000000E+00} \\
u_3^\star &= \texttt{1.0000000000000000E+00} \\
u_4^\star &= \texttt{0.0000000000000000E+00} \\
u_5^\star &= \texttt{0.0000000000000000E+00}
\end{aligned}
$$

The multipliers are not uniquely determined, but $u_3 = 1 + 2u_2$, so I arbitrarily chose $u_2^\star = 0$.

Provenance
Floudas [15, §9.2.3] ← Liu [21, p166]. The location of the nonnegativity constraint on x is ambiguous in [15] but stated in [21], resulting in the inducible region shown.

Test Problem 2

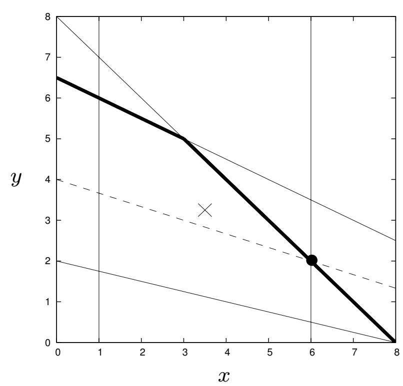
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x + 3y$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + 1 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 6 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = y + x - 8 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -4y - x + 8 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 2y + x - 13 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	6	6.0000000000000000E+00	6.1004915073699495E+00
x^L	1	1.0000000000000000E+00	5.9004915073699502E+00
y^H	6	6.0000000000000000E+00	2.1098691460679877D+00
y^L	1/2	5.0000000000000000E-01	1.9098691460679877D+00

Solution



$$y^\star(x) = \begin{cases} \frac{1}{2}(13 - x) & x \le 3 \\ 8 - x & x \ge 3 \end{cases}$$

$x^\star$ = 6.0000000000000000E+00
$y^\star$ = 2.0000000000000000E+00
$f_0^\star$ = 1.2000000000000000E+01
$u_1^\star$ = 1.0000000000000000E+00
$u_2^\star$ = 0.0000000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00

Provenance

Dempe [9, p22]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 3

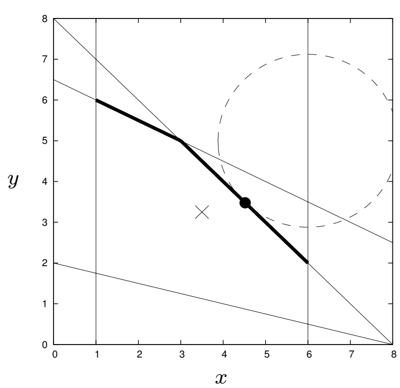
$$
\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-6)^2 + (y-5)^2
$$

$$
\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + 1 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 6 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = y + x - 8 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -4y - x + 8 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 2y + x - 13 \le 0 \end{cases} \end{array} \right] \end{cases}
$$

Starting Bounds

	natural	fair	tight
x^H	6	6.0000000000000000E+00	4.6004915073699495E+00
x^L	1	1.0000000000000000E+00	4.4004915073699502E+00
y^H	6	6.0000000000000000E+00	3.6098691460679877E+00
y^L	1/2	5.0000000000000000E-01	3.4098691460679875E+00

Solution



$$
y^\star(x) = \begin{cases} \frac{1}{2}(13-x) & 0 \le x \le 3 \\ 8 - x & 3 \le x \le 8 \end{cases}
$$

$$
\begin{aligned}
x^\star &= 4.5000000000000000\text{E+00} \\
y^\star &= 3.5000000000000000\text{E+00} \\
f_0^\star &= 4.5000000000000000\text{E+00} \\
u_1^\star &= 1.0000000000000000\text{E+00} \\
u_2^\star &= 0.0000000000000000\text{E+00} \\
u_3^\star &= 0.0000000000000000\text{E+00}
\end{aligned}
$$

Provenance

Dempe [9, p22], but with the nonlinear outer objective given above. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 4

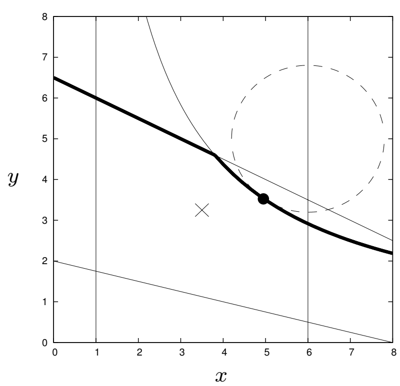
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-6)^2 + (y-5)^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + 1 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 6 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = xy - \frac{35}{2} \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -4y - x + 8 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 2y + x - 13 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	6	6.0000000000000000E+00	5.0541082948053804E+00
x^L	1	1.0000000000000000E+00	4.8541082948053811E+00
y^H	6	6.0000000000000000E+00	3.6426414114535057D+00
y^L	1/2	5.0000000000000000E-01	3.4426414114535056E+00

Solution



$$y^\star(x) = \begin{cases} \frac{1}{2}(13-x) & x \le \frac{1}{2}(13-\sqrt{29}) \\ \frac{1}{2}(35/x) & x \ge \frac{1}{2}(13-\sqrt{29}) \end{cases}$$

$$\begin{aligned} x^\star &= \texttt{4.9536167874354312E+00} \\ y^\star &= \texttt{3.5327722653855180E+00} \\ f_0^\star &= \texttt{3.2476750527588929E+00} \\ u_1^\star &= \texttt{2.0187270087917245E-01} \\ u_2^\star &= \texttt{0.0000000000000000E+00} \\ u_3^\star &= \texttt{0.0000000000000000E+00} \end{aligned}$$

Provenance

Dempe [9, p22], but with the nonlinear f_0 and g_1 given above. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 5

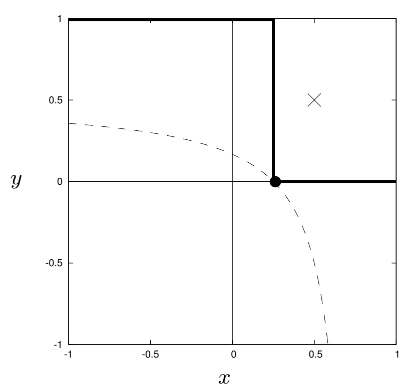
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -(4x-3)y + (2x+1)$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \leq 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 1 \leq 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -(1-4x)y - (2x+2) \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -y \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = y - 1 \leq 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	1	1.0000000000000000E+00	2.6004915073699497E-01
x^L	0	0.0000000000000000E+00	2.4004915073699498E-01
y^H	1	1.0000000000000000E+00	1.0986914606798769E-01
y^L	0	0.0000000000000000E+00	-9.0130853932012325D-02

Solution



$$y^\star(x) = \begin{cases} 1 & x \leq \frac{1}{4} \\ 0 & x \geq \frac{1}{4} \end{cases}$$

$x^\star$ = 2.5000000000000000E-01
$y^\star$ = 0.0000000000000000E+00
$f_0^\star$ = 1.5000000000000000E+00
$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 0.0000000000000000E+00

Provenance

Floudas [15, §9.3.9] ← Yezza [27, p198] ← Bard [6, §3.1]. The location of the bound constraints on x is ambiguous in [15] but stated in [27], resulting in the inducible region shown.

Test Problem 6

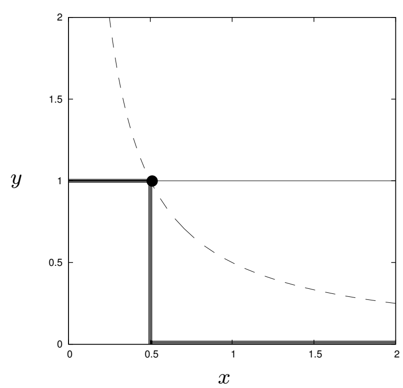
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -xy$$
$$\text{subject to} \left\{ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min\limits_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (x-\frac{1}{2})y \\ \text{subject to} \left\{ \begin{array}{l} g_1(\mathbf{y};\mathbf{x}) = y - 1 \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y \leq 0 \end{array} \right. \end{array} \right] \right.$$

Starting Bounds

	natural	fair	tight
x^H	$+\infty$	5.5000000000000000E+00	5.1004915073699497E-01
x^L	0	0.0000000000000000E+00	4.9004915073699501E-01
y^H	1	1.0000000000000000E+00	1.1098691460679877E+00
y^L	0	0.0000000000000000E+00	9.0986914606798763E-01

We are minimizing $-xy$ and $y \geq 0$, so it does not make sense to consider negative values of x and the natural bound x^L is zero by inspection.

Solution



$$y^\star(x) = \left\{ \begin{array}{ll} 1 & x \leq \frac{1}{2} \\ \in [0,1] & x = \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \end{array} \right.$$

$$\begin{array}{lcl} x^\star & = & 5.0000000000000000E-01 \\ y^\star & = & 1.0000000000000000E+00 \\ f_0^\star & = & -5.0000000000000000E-01 \\ u_1^\star & = & 0.0000000000000000E+00 \\ u_2^\star & = & 0.0000000000000000E+00 \end{array}$$

The starting point is outside the frame of this picture.

Provenance

Dempe [9, p236]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

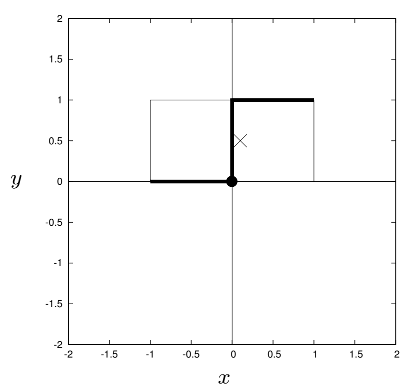
Test Problem 7

$$
\begin{aligned}
&\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x^2 + y^2 \\
&\text{subject to} \begin{cases}
f_1(\mathbf{x},\mathbf{y}) = x - 1 \le 0 \\
f_2(\mathbf{x},\mathbf{y}) = -x - 1 \le 0 \\
\mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -xy \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = y - 1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y \le 0 \end{cases} \end{array} \right]
\end{cases}
\end{aligned}
$$

Starting Bounds

	natural	fair	tight
x^H	1	1.2000000000000000E+00	1.0049150736994984E-01
x^L	-1	-1.0000000000000000E+00	-9.9508492630050169E-02
y^H	1	1.0000000000000000E+00	1.0986914606798769E-01
y^L	0	0.0000000000000000E+00	-9.0130853932012325E-02

Solution



$$
y^\star(x) = \begin{cases} 0 & -1 \le x \le 0 \\ \in [0,1] & x = 0 \\ 1 & 0 \le x \le 1 \end{cases}
$$

$$
\begin{aligned}
x^\star &= \texttt{0.0000000000000000E+00} \\
y^\star &= \texttt{0.0000000000000000E+00} \\
f_0^\star &= \texttt{0.0000000000000000E+00} \\
u_1^\star &= \texttt{0.0000000000000000E+00} \\
u_2^\star &= \texttt{0.0000000000000000E+00}
\end{aligned}
$$

The optimal objective contour for this problem is the single point $[x^\star, y^\star]^\top$ so it does not show dashed in the picture.

Provenance

Dempe [9, p121]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 8

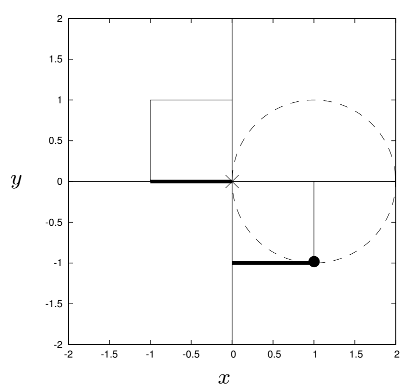
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-1)^2 + y^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x - 1 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x - 1 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = y - 1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y - 1 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = xy \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	1	1.0000000000000000E+00	1.1004915073699499E+00
x^L	-1	-1.0000000000000000E+00	9.0049150736994987E-01
y^H	1	1.0000000000000000E+00	-8.9013085393201230E-01
y^L	-1	-1.0000000000000000E+00	-1.0901308539320123E+00

Solution



$$y^\star(x) = \begin{cases} 0 & -1 \le x < 0 \\ -1 & 0 \le x \le 1 \end{cases}$$

$x^\star$ = 1.0000000000000000E+00
$y^\star$ = -1.0000000000000000E+00
$f_0^\star$ = 1.0000000000000000E+00
$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 1.0000000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00

The inducible region is disconnected and does not include the origin, so the origin is not an optimal point for this bilevel program.

Provenance

Luo [22, §1.1.2]. No outer objective is given in [22], so I selected the one given above.

Test Problem 9

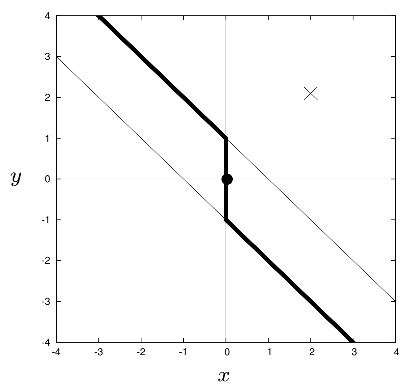
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (y-x)^2 + x^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x - 20 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x - 20 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = xy \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -x - y - 1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = x + y - 1 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	20	2.4000000000000000E+01	1.0049150736994984E-01
x^L	-20	-2.0000000000000000E+01	-9.9508492630050169E-02
y^H	21	2.5199999999999999E+01	1.0986914606798769E-01
y^L	-21	-2.1000000000000000E+01	-9.0130853932012325E-02

Solution



$$y^\star(x) = \begin{cases} 1-x & x \le 0 \\ \in [-1,1] & x = 0 \\ -1-x & x \ge 0 \end{cases}$$

$$\begin{aligned} x^\star &= 0.0000000000000000E+00 \\ y^\star &= 0.0000000000000000E+00 \\ f_0^\star &= 0.0000000000000000E+00 \\ u_1^\star &= 0.0000000000000000E+00 \\ u_2^\star &= 0.0000000000000000E+00 \end{aligned}$$

For clarity only the central region of the complete contour plot is shown above, so the bounds constraints on x are outside the frame of the picture. The optimal objective contour for this problem is the single point $[x^\star, y^\star]^\top$ so it does not show dashed in the picture.

Provenance

Dempe [9, p226,p233]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 10

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-8)^2 + (y-7)^2$$

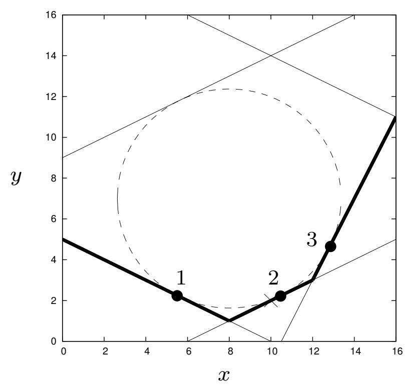
$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -x - 2y + 10 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = x - 2y - 6 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 2x - y - 21 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = x + 2y - 38 \le 0 \\ g_5(\mathbf{y};\mathbf{x}) = -x + 2y - 18 \le 0 \\ g_6(\mathbf{y};\mathbf{x}) = -y \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	16	1.2000000000000000E+01	1.1404915073699499E+01
x^L	0	8.0000000000000000E+00	9.4049150736994989E+00
y^H	14	3.0000000000000000E+00	2.3098691460679879E+00
y^L	1	1.0000000000000000E+00	2.1098691460679877E+00

The fair bounds were chosen to bracket solution 2.

Solution



$$\begin{aligned} x_1^\star &= 5.6000000000000000E+00 \\ y_1^\star &= 2.2000000000000000E+00 \\ u_2^\star &= 0.0000000000000000E+00 \\ u_3^\star &= 1.0000000000000000E+00 \\ x_2^\star &= 1.0400000000000000E+01 \\ y_2^\star &= 2.2000000000000000E+00 \\ u_1^\star &= 0.0000000000000000E+00 \\ u_3^\star &= 1.0000000000000000E+00 \\ x_3^\star &= 1.2800000000000000E+01 \\ y_3^\star &= 4.6000000000000000E+00 \\ u_1^\star &= 0.0000000000000000E+00 \\ u_2^\star &= 5.0000000000000000E-01 \\ f_0^\star &= 2.8800000000000000E+01 \end{aligned}$$

$$y^\star(x) = \begin{cases} 5 - \frac{1}{2}x & 0 \le x \le 8 \\ -3 + \frac{1}{2}x & 8 \le x \le 12 \\ -21 + 2x & 12 \le x \le 16 \end{cases}$$

There are three alternate optima as shown. For each, the KKT multipliers that are not listed are zero. The nonzero multipliers are related but not uniquely determined.

Provenance
Luo [22, §1.1.1]. No outer objective is given in [22], so I selected the one given above.

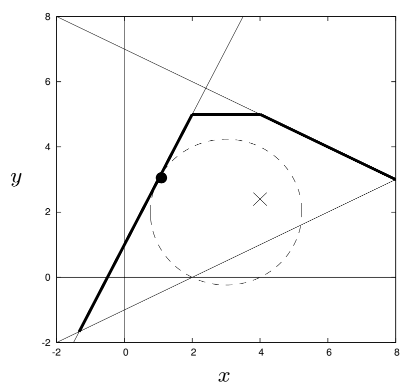
Test Problem 11

$$
\begin{aligned}
&\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-3)^2 + (y-2)^2 \\
&\text{subject to} \begin{cases}
f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\
f_2(\mathbf{x},\mathbf{y}) = x - 8 \le 0 \\
\mathbf{y} \text{ solves} \left[\begin{aligned}
&\min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-5)^2 \\
&\text{subject to} \begin{cases}
g_1(\mathbf{y};\mathbf{x}) = -2x + y - 1 \le 0 \\
g_2(\mathbf{y};\mathbf{x}) = x - 2y - 2 \le 0 \\
g_3(\mathbf{y};\mathbf{x}) = x + 2y - 14 \le 0
\end{cases}
\end{aligned} \right]
\end{cases}
\end{aligned}
$$

Starting Bounds

	natural	fair	tight
x^H	8	8.0000000000000000E+00	1.1004915073699499E+00
x^L	0	0.0000000000000000E+00	9.0049150736994987E-01
y^H	29/5	5.7999999999999998E+00	3.1098691460679877E+00
y^L	-1	-1.0000000000000000E+00	2.9098691460679875E+00

Solution



$$
y^\star(x) = \begin{cases}
2x+1 & -\frac{4}{3} \le x \le 2 \\
5 & 2 \le x \le 4 \\
7 - \frac{1}{2}x & 4 \le x \le 8
\end{cases}
$$

$x^\star$ = 1.0000000000000000E+00
$y^\star$ = 3.0000000000000000E+00
$f_0^\star$ = 5.0000000000000000E+00

$u_1^\star$ = 4.0000000000000000E+00
$u_2^\star$ = 0.0000000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00

The central segment of the inducible region shown above is the result of the inner optimization, rather than being part of a constraint contour.

Provenance

Floudas [15, §9.3.6] ⇜ Clark [7, p90]. The source of this problem is given in Floudas [15] as Clark [7]. However, [7] has $g_2(\mathbf{y};\mathbf{x}) = x - 2y + 2 \le 0$, so the problems are similar but not the same. The location of the bounds on x is ambiguous in [15] but stated in [7], resulting in the inducible region shown.

Test Problem 12

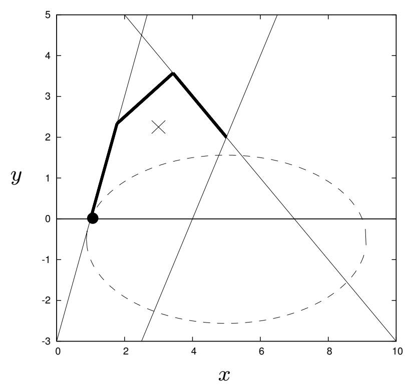
$$
\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-5)^2 + (2y+1)^2
$$

$$
\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-1)^2 - \frac{3}{2}xy \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -3x + y + 3 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = x - \frac{1}{2}y - 4 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = x + y - 7 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y \le 0 \end{cases} \end{array} \right] \end{cases}
$$

Starting Bounds

	natural	fair	tight
x^H	5	5.0000000000000000E+00	1.1004915073699499E+00
x^L	1	1.0000000000000000E+00	9.0049150736994987E-01
y^H	9/2	4.5000000000000000E+00	1.0986914606798769E-01
y^L	0	0.0000000000000000E+00	-9.0130853932012325E-02

Solution



$$
y^\star(x) = \begin{cases} 3x - 3 & 1 \le x \le \frac{16}{9} \\ \frac{3}{4}x + 1 & \frac{16}{9} \le x \le \frac{24}{7} \\ 7 - x & \frac{24}{7} \le x \le 5 \end{cases}
$$

$$
\begin{aligned}
x^\star &= 1.0000000000000000\text{E+00} \\
y^\star &= 0.0000000000000000\text{E+00} \\
f_0^\star &= 1.7000000000000000\text{E+01} \\
u_1^\star &= 5.0000000000000000\text{E-01} \\
u_2^\star &= 0.0000000000000000\text{E+00} \\
u_3^\star &= 0.0000000000000000\text{E+00} \\
u_4^\star &= 0.0000000000000000\text{E+00}
\end{aligned}
$$

The central segment of the inducible region shown above is the result of the inner optimization, rather than being part of a constraint contour. The multipliers are not uniquely determined, but $u_4 = u_1 - \frac{1}{2}$, so I arbitrarily chose $u_1^\star = \frac{1}{2}$.

Provenance

Floudas [15, §9.3.2] ← Shimizu [24, p430] ← Bard [3, p18-19]. The location of the nonnegativity constraints is ambiguous in [15] but stated in [24] and [3], resulting in the inducible region shown. Also see Floudas [15, §9.3.8] ← Visweswaran [25, p158] ← Bard [3, p18-19].

Test Problem 13

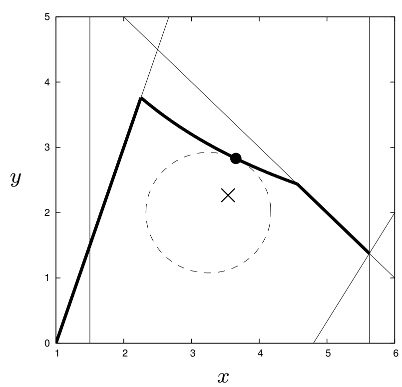
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x - \tfrac{13}{4})^2 + (y-2)^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + \frac{3}{2} \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - \frac{45}{8} \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-8)^2 + \frac{1}{2}xy^2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -3x + y + 3 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = \frac{5}{3}x - y - 8 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = x + y - 7 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	45/8	5.6250000000000000E+00	3.7626191926881543E+00
x^L	3/2	1.5000000000000000E+00	3.5626191926881541E+00
y^H	9/2	4.5000000000000000E+00	2.9356620086146439E+00
y^L	0	0.0000000000000000E+00	2.7356620086146437E+00

Solution



$$y^\star(x) = \begin{cases} 3x-3 & 1 \le x \le \sqrt{\frac{91}{12}} - \frac{1}{2} \\ 16/(2+x) & \sqrt{\frac{91}{12}} - \frac{1}{2} \le x \le \frac{1}{2}(5+\sqrt{17}) \\ 7-x & \frac{1}{2}(5+\sqrt{17}) \le x \le \frac{45}{8} \end{cases}$$

$$\begin{aligned} x^\star &= \texttt{3.6621276853182042E+00} \\ y^\star &= \texttt{2.8257928625466561E+00} \\ f_0^\star &= \texttt{8.5178308083874112E-01} \\ u_1^\star &= \texttt{0.0000000000000000E+00} \\ u_2^\star &= \texttt{0.0000000000000000E+00} \\ u_3^\star &= \texttt{0.0000000000000000E+00} \\ u_4^\star &= \texttt{0.0000000000000000E+00} \end{aligned}$$

The curved portion of the inducible region shown above is the result of the inner optimization, rather than being a constraint contour, and the optimal point is interior to the feasible region of the inner problem.

Provenance
This problem, due to Rugenstein, is a modification of one given by Bard [3, p18-19].

Test Problem 14

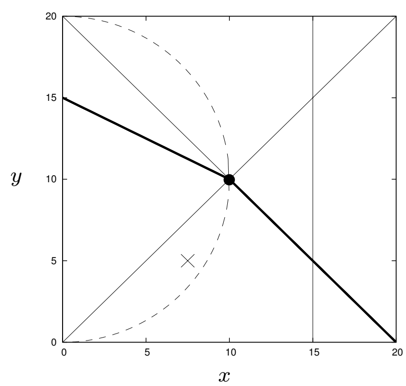
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x^2 + (y-10)^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x + y \leq 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 15 \leq 0 \\ f_3(\mathbf{x},\mathbf{y}) = -x \leq 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (x + 2y - 30)^2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = x + y - 20 \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y \leq 0 \\ g_3(\mathbf{y};\mathbf{x}) = y - 20 \leq 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	15	1.5000000000000000E+01	1.1004915073699499E+01
x^L	0	0.0000000000000000E+00	9.0049150736994985E+00
y^H	20	1.0000000000000000E+01	1.1098691460679877E+01
y^L	0	0.0000000000000000E+00	9.0986914606798770E+00

Solution



$$y^\star(x) = \begin{cases} 15 - \frac{1}{2}x & 0 \leq x \leq 10 \\ 20 - x & 10 \leq x \leq 20 \end{cases}$$

$$\begin{aligned} x^\star &= 1.0000000000000000\text{E+01} \\ y^\star &= 1.0000000000000000\text{E+01} \\ f_0^\star &= 1.0000000000000000\text{E+02} \\ u_1^\star &= 0.0000000000000000\text{E+00} \\ u_2^\star &= 0.0000000000000000\text{E+00} \\ u_3^\star &= 0.0000000000000000\text{E+00} \end{aligned}$$

The left segment of the inducible region shown above is the result of the inner optimization, rather than being part of a constraint contour.

Provenance

Floudas [15, §9.3.3] ← Visweswaran [25, p159] ← Shimizu [23, p465]. The location of the bounds on x is ambiguous in [15] but stated in [25] and [23], resulting in the inducible region shown.

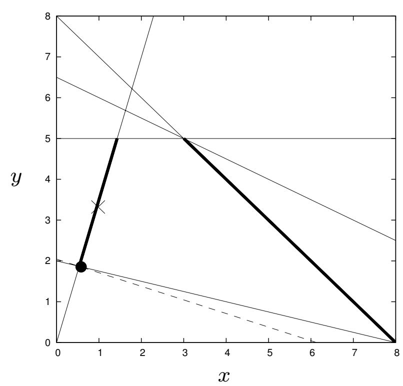
Test Problem 15

$$
\begin{aligned}
&\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x + 3y \\
&\text{subject to} \left\{
\begin{array}{l}
f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\
f_2(\mathbf{x},\mathbf{y}) = x - 8 \le 0 \\
f_3(\mathbf{x},\mathbf{y}) = y - 5 \le 0 \\
\mathbf{y} \text{ solves} \left[
\begin{array}{l}
\min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -y \\
\text{subject to} \left\{
\begin{array}{l}
g_1(\mathbf{y};\mathbf{x}) = x + y - 8 \le 0 \\
g_2(\mathbf{y};\mathbf{x}) = -x - 4y + 8 \le 0 \\
g_3(\mathbf{y};\mathbf{x}) = x + 2y - 13 \le 0 \\
g_4(\mathbf{y};\mathbf{x}) = -7x + 2y \le 0
\end{array}
\right.
\end{array}
\right]
\end{array}
\right.
\end{aligned}
$$

Starting Bounds

	natural	fair	tight
x^H	8	1.4285714285714288E+00	5.4338248407032830E-01
x^L	8/15	5.3333333333333321E-01	5.2338248407032828E-01
y^H	5	5.0000000000000000E+00	1.9765358127346544E+00
y^L	0	1.6428571428571423E+00	1.7765358127346544E+00

Solution



$$
y^\star(x) = \left\{
\begin{array}{ll}
\frac{7}{2}x & \frac{8}{15} \le x \le \frac{10}{7} \\
8 - x & 3 \le x \le 8
\end{array}
\right.
$$

$$
\begin{aligned}
x^\star &= 5.3333333333333333\text{E-}01 \\
y^\star &= 1.8666666666666667\text{E+}00 \\
f_0^\star &= 6.1333333333333333\text{E+}00 \\
u_1^\star &= 0.0000000000000000\text{E+}00 \\
u_2^\star &= 0.0000000000000000\text{E+}00 \\
u_3^\star &= 0.0000000000000000\text{E+}00 \\
u_4^\star &= 5.0000000000000000\text{E-}01
\end{aligned}
$$

The outer constraint on y causes the inducible region to be disconnected as shown. The multipliers are not uniquely determined, but $u_4 = 2u_2 + \frac{1}{2}$ so I arbitrarily chose $u_2^\star = 0$.

Provenance

Dempe [9, p25]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 16

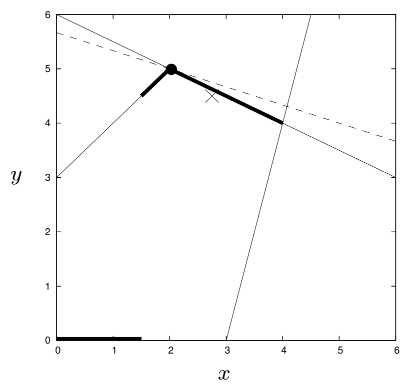
$$
\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -x - 3y
$$

$$
\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -(y - \frac{9}{4})^2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -y \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -x + y - 3 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = x + 2y - 12 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = 4x - y - 12 \le 0 \end{cases} \end{array} \right] \end{cases}
$$

Starting Bounds

	natural	fair	tight
x^H	4	4.0000000000000000E+00	2.1004915073699499E+00
x^L	0	1.5000000000000000E+00	1.9004915073699498E+00
y^H	5	5.0000000000000000E+00	5.1098691460679877E+00
y^L	0	4.0000000000000000E+00	4.9098691460679875E+00

Solution



$$
y^\star(x) = \begin{cases} 0 & 0 \le x \le \frac{3}{2} \\ 3 + x & \frac{3}{2} \le x \le 2 \\ -\frac{1}{2}x + 6 & 2 \le x \le 4 \end{cases}
$$

$$
\begin{aligned}
x^\star &= \texttt{2.0000000000000000E+00} \\
y^\star &= \texttt{5.0000000000000000E+00} \\
f_0^\star &= \texttt{-1.7000000000000000E+01} \\
u_1^\star &= \texttt{0.0000000000000000E+00} \\
u_2^\star &= \texttt{1.8333333333333333E+00} \\
u_3^\star &= \texttt{1.8333333333333333E+00} \\
u_4^\star &= \texttt{0.0000000000000000E+00}
\end{aligned}
$$

The inner optimization causes the inducible region to be disconnected as shown. The KKT multipliers are not uniquely determined, but $u_2 + 2u_3 = \frac{11}{2}$ so I arbitrarily chose $u_2^\star = u_3^\star = \frac{11}{6}$.

Provenance

This problem, due to Rugenstein, was contrived for the original version of this paper.

Test Problem 17

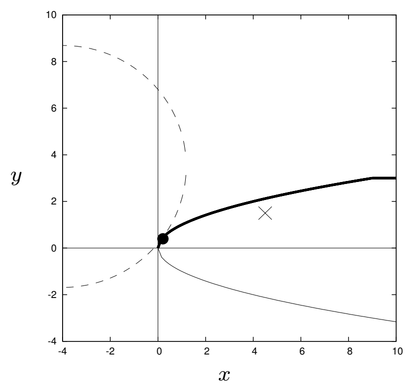
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x+4)^2 + (y-3.5)^2$$

$$\text{subject to} \left\{ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-3)^2 \\ \text{subject to} \left\{ g_1(\mathbf{y};\mathbf{x}) = -x + y^2 \le 0 \right. \end{array} \right] \right.$$

Starting Bounds

	natural	fair	tight
x^H	9	9.0000000000000000E+00	1.5216550371274204E-01
x^L	0	0.0000000000000000E+00	1.3216550371274202E-01
y^H	3	3.0000000000000000E+00	3.8797014099193522E-01
y^L	0	0.0000000000000000E+00	3.6797014099193520E-01

Solution



$$y^\star(x) = \begin{cases} +\sqrt{x} & 0 \le x \le 9 \\ 3 & x \ge 9 \end{cases}$$

$x^\star$ = 1.4211635297574703E-01
$y^\star$ = 3.7698322638513644E-01
$f_0^\star$ = 2.6910361649868895E+01
$u^\star$ = 6.9579137479584235E+00

Provenance

Dempe [9, p130]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 18

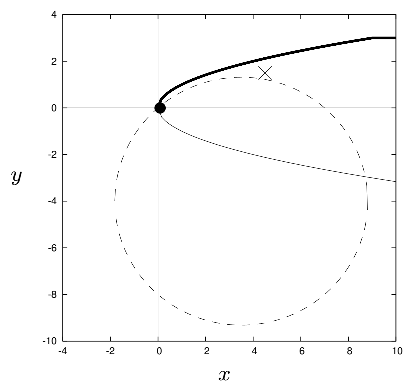
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x-\tfrac{7}{2})^2 + (y+4)^2$$

$$\text{subject to} \left\{ \; \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y-3)^2 \\ \text{subject to} \left\{ \; g_1(\mathbf{y};\mathbf{x}) = -x + y^2 \le 0 \right. \end{array} \right] \right.$$

Starting Bounds

	natural	fair	tight
x^H	9	9.0000000000000000E+00	1.0049150736994984E-01
x^L	0	0.0000000000000000E+00	-9.9508492630050169E-02
y^H	3	3.0000000000000000E+00	1.0986914606798769E-01
y^L	0	0.0000000000000000E+00	-9.0130853932012325E-02

Solution



$$y^\star(x) = \left\{ \begin{array}{ll} +\sqrt{x} & 0 \le x \le 9 \\ 3 & x \ge 9 \end{array} \right.$$

$$\begin{array}{lcl} x^\star & = & \texttt{0.0000000000000000E+00} \\ y^\star & = & \texttt{0.0000000000000000E+00} \\ f_0^\star & = & \texttt{2.8250000000000000E+01} \\ u^\star & = & +\infty \end{array}$$

The KKT conditions for the inner problem with x fixed require $u = \frac{3}{y} - 1$.

Provenance

Luo [22, p354] ← Dempe [8, p351]. In [22], x is used (as here) for the outer variable but z is used for the inner variable. The solution $[1.7296, 1.3151]^\top$ given in [22] is in the inducible region but yields $f_0 \approx 31.385$ and is not optimal. In [8], x is used for the inner variable and y for the outer (opposite to here).

Test Problem 19

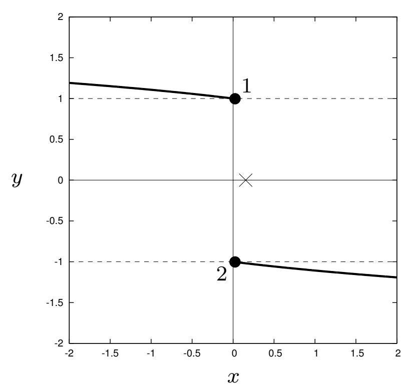
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = y^2$$
$$\text{subject to} \left\{ \ \mathbf{y} \text{ solves} \left[\ \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y^4 - 2y^2 + xy + 1 \ \right] \right.$$

Starting Bounds

	natural	fair	tight
x^H	$8\sqrt{3}/9$	1.8475208614068026E+00	1.0049150736994984E-01
x^L	$-8\sqrt{3}/9$	-1.5396007178390021E+00	-9.9508492630050169D-02
y^H	$2\sqrt{3}/3$	1.1547005383792515E+00	1.1098691460679877E+00
y^L	$-2\sqrt{3}/3$	-1.1547005383792515E+00	9.0986914606798763E-01

The natural bounds stated for x are the values where the locus of points $(x, y : dg_0(y;x)/dy = 0)$ reverses direction (on a Z-shaped segment that is omitted from the graph below because its points are not minimizers of $g_0(y;x)$). The natural bounds stated for y correspond to those values of x. The fair bounds were chosen to bracket solution 1.

Solution



$$f_0^\star = 1.0000000000000000E+00$$

$$x_1^\star = 0.0000000000000000E+00$$
$$y_1^\star = 1.0000000000000000E+00$$

$$x_2^\star = 0.0000000000000000E+00$$
$$y_2^\star = -1.0000000000000000E+00$$

$$y^\star(x) = \begin{cases} +\dfrac{\left(\sqrt[3]{-27x+3\sqrt{81x^2-192}}\right)^2+12}{6\sqrt[3]{-27x+3\sqrt{81x^2-192}}} & x \le 0 \\ -\dfrac{\left(\sqrt[3]{+27x+3\sqrt{81x^2-192}}\right)^2+12}{6\sqrt[3]{+27x+3\sqrt{81x^2-192}}} & x \ge 0 \end{cases}$$

There are two optima as shown.

Provenance

Dempe [9, p168]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 20

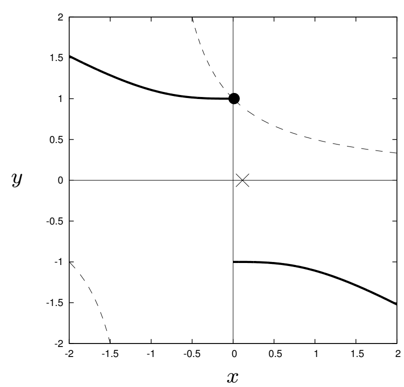
$$\begin{array}{l}\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -(x+1)y \\ \text{subject to} \left\{ \ \mathbf{y} \text{ solves} \left[\ \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y^4 - 2y^2 + x^3 y + 1 \ \right] \right.\end{array}$$

Starting Bounds

	natural	fair	tight
x^H	$2\sqrt{3}/3$	1.3856406460551018E+00	1.0049150736994984E-01
x^L	$-2\sqrt{3}/3$	-1.1547005383792515E+00	-9.9508492630050169E-02
y^H	$2\sqrt{3}/3$	1.1547005383792515E+00	1.1098691460679877E-01
y^L	$-2\sqrt{3}/3$	-1.1547005383792515E+00	9.0986914606798763E-01

The natural bounds stated for x are the values where the locus of points $(x, y : dg_0(y;x)/dy = 0)$ reverses direction (on a Z-shaped segment that is omitted from the graph below because its points are not minimizers of $g_0(y;x)$). The natural bounds stated for y correspond to those values of x.

Solution



$$\begin{array}{lcl} f_0^\star & = & \text{-1.0000000000000000E+00} \\ x^\star & = & \text{0.0000000000000000E+00} \\ y^\star & = & \text{1.0000000000000000E+00} \end{array}$$

$$y^\star(x) = \begin{cases} +\dfrac{\left(\sqrt[3]{-27x^3+3\sqrt{81x^6-192}}\right)^2+12}{6\sqrt[3]{-27x^3+3\sqrt{81x^6-192}}} & x \le 0 \\ -\dfrac{\left(\sqrt[3]{+27x^3+3\sqrt{81x^6-192}}\right)^2+12}{6\sqrt[3]{+27x^3+3\sqrt{81x^6-192}}} & x \ge 0 \end{cases}$$

Provenance

Dempe [9, p169]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here. Dempe specifies an outer objective of $-xy$, but that yields a problem with infima rather than a minimizing point. To provide the problem with a minimizing point, $f_0(\mathbf{x},\mathbf{y})$ was changed to the function given above.

Test Problem 21

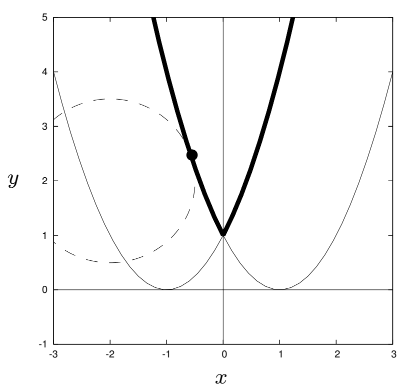
$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = (x+2)^2 + (y-2)^2$$

$$\text{subject to} \left\{ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = y^2 \\ \text{subject to} \left\{ \begin{array}{l} g_1(\mathbf{y};\mathbf{x}) = (x-1)^2 - y \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = (x+1)^2 - y \le 0 \end{array} \right. \end{array} \right] \right.$$

Starting Bounds

	natural	fair	tight
x^H	$+\infty$	-5.6746837485242008E-02	-5.5741922411542710E-01
x^L	$-\infty$	-5.6746837485242203E+00	-5.7741922411542712E-01
y^H	$+\infty$	1.7026528167787426E+01	2.5668262522304808E+00
y^L	1	1.0000000000000000E+00	2.3668262522304806E+00

Solution



$$y^\star(x) = \left\{ \begin{array}{ll} (x-1)^2 & x \le 0 \\ (x+1)^2 & x \ge 0 \end{array} \right.$$

$$\begin{array}{lcl} x^\star & = & \texttt{-5.6746837485242209E-01} \\ y^\star & = & \texttt{2.4569571061624932E+00} \\ f_0^\star & = & \texttt{2.2609566539203607E+00} \\ u_1^\star & = & \texttt{4.9139142123249864E+00} \\ u_2^\star & = & \texttt{0.0000000000000000E+00} \end{array}$$

The starting point is outside the frame of this picture.

Provenance

Dempe [9, p196]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

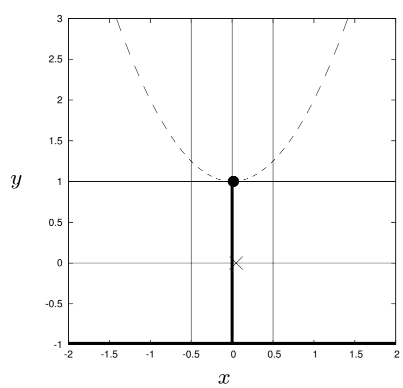
Test Problem 22

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -y + x^2$$
$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x - \frac{1}{2} \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x - \frac{1}{2} \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = x^2 y \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = y - 1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y - 1 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	$1/2$	5.9999999999999998E-01	1.0049150736994984E-01
x^L	$-1/2$	-5.0000000000000000E-01	-9.9508492630050169D-02
y^H	1	1.0000000000000000E+00	1.1098691460679877E+00
y^L	-1	-1.0000000000000000E+00	9.0986914606798763E-01

Solution



$$y^\star(x) = \begin{cases} -1 & \forall x \\ \in [-1,1] & x = 0 \end{cases}$$

$$\begin{aligned} x^\star &= \texttt{0.0000000000000000E+00} \\ y^\star &= \texttt{1.0000000000000000E+00} \\ f_0^\star &= \texttt{-1.0000000000000000E+00} \\ u_1^\star &= \texttt{0.0000000000000000E+00} \\ u_2^\star &= \texttt{0.0000000000000000E+00} \end{aligned}$$

Provenance

Dempe [9, p227]. Dempe uses y for the outer variable and x for the inner, so his notation is opposite that used here.

Test Problem 23

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x + y_2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \leq 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = 2y_1 + xy_2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = x - y_1 - y_2 \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y_1 \leq 0 \\ g_3(\mathbf{y};\mathbf{x}) = -y_2 \leq 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	$+\infty$	1.0000000000000000E+00	1.0420317017925204E-01
x^L	0	0.0000000000000000E+00	-9.5796829820747970E-02
y_1^H	$+\infty$	1.0000000000000000E+00	1.0986914606798769E-01
y_1^L	0	0.0000000000000000E+00	-9.0130853932012325E-02
y_2^H	$+\infty$	1.0000000000000000E+00	1.0049150736994984E-01
y_2^L	0	0.0000000000000000E+00	-9.9508492630050169E-02

Solution

$x^\star$ = 0.0000000000000000E+00
$y_1^\star$ = 0.0000000000000000E+00
$y_2^\star$ = 0.0000000000000000E+00
$f_0^\star$ = 0.0000000000000000E+00

$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 2.0000000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00

The multipliers are not uniquely determined, but $u_2 = 2 - u_1$ and $u_3 = u_1$, so I arbitrarily chose $u_1^\star = 0$.

Provenance

Floudas [15, §9.3.10] ⇜ Bard [4, p373]. The source of this problem is given in Floudas [15] as Bard [4]. However, [4] has the outer constraints $2 \leq x \leq 4$ and the inner constraint $g_1(\mathbf{y};\mathbf{x}) = -x + y_1 + y_2 \geq 4$, and it leaves $\mathbf{y}$ unconstrained, so the problems are not the same. The location of the bounds on x is ambiguous in [15] but stated in [4]. The location of the extra bounds on $\mathbf{y}$ is also ambiguous in [15], so I assumed they are inner constraints.

Test Problem 24

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -x - 3y_1 + 2y_2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = -x \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = x - 8 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = -y_1 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -y_1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = y_1 - 4 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = -2x + y_1 + 4y_2 - 16 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = 8x + 3y_1 - 2y_2 - 48 \le 0 \\ g_5(\mathbf{y};\mathbf{x}) = -2x + y_1 - 3y_2 + 12 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x^H	8	8.0000000000000000E+00	5.1042031701792521E+00
x^L	0	0.0000000000000000E+00	4.9042031701792519E+00
y_1^H	4	4.0000000000000000E+00	4.1098691460679877E+00
y_1^L	0	0.0000000000000000E+00	3.9098691460679875E+00
y_2^H	8	8.0000000000000000E+00	2.1004915073699499E+00
y_2^L	$-4/3$	-1.3333333333333339E+00	1.9004915073699498E+00

Solution

$$\begin{aligned} x^\star &= 5.0000000000000000E{+}00 \\ y_1^\star &= 4.0000000000000000E{+}00 \\ y_2^\star &= 2.0000000000000000E{+}00 \\ f_0^\star &= -1.3000000000000000E{+}01 \end{aligned} \qquad \begin{aligned} u_1^\star &= 0.0000000000000000E{+}00 \\ u_2^\star &= 1.0000000000000000E{+}00 \\ u_3^\star &= 0.0000000000000000E{+}00 \\ u_4^\star &= 0.0000000000000000E{+}00 \\ u_5^\star &= 0.0000000000000000E{+}00 \end{aligned}$$

Provenance

Floudas [15, §9.2.2] ⇜ Clark [7, p89]. The location of the bounds on x is ambiguous in [15] but given in [7]. In [15], the bounds on y_1 are included as explicit constraints in the inner problem, but bounds on the vector $\mathbf{y}$ are also separately given as $\mathbf{0} \le \mathbf{y} \le \mathbf{4}$. This constrains y_2 as well, whereas [7] leaves y_2 unconstrained. It is ambiguous in [15] whether this added constraint is intended for the inner or outer problem, but either way it affects the inducible region and makes the problem different from the one given in [7]. In this case I assumed that the bounds on y_2 stated in [15] were given in error, and used the formulation of [7] in which y_2 is unconstrained.

Test Problem 25

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -2x_1 + x_2 + \tfrac{1}{2}y_1$$

$$\text{subject to} \begin{cases} f_1(\mathbf{y};\mathbf{x}) = 2x_1 + x_2 - 2 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x_1 \le 0 \\ f_3(\mathbf{x},\mathbf{y}) = -x_2 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = x_1 + x_2 - 4y_1 + y_2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -2x_1 + y_1 - y_2 + 5/2 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = x_1 - 3x_2 + y_2 - 2 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = -y_1 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y_2 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x_1^H	1	1.0000000000000000E+00	1.1042031701792521E+00
x_1^L	0	0.0000000000000000E+00	9.0420317017925200E-01
x_2^H	2	2.0000000000000000E+00	1.0145616531775192E-01
x_2^L	0	0.0000000000000000E+00	-9.8543834682248088D-02
y_1^H	11/2	5.5000000000000000E+00	5.1098691460679879E-01
y_1^L	0	0.0000000000000000E+00	4.9098691460679877E-01
y_2^H	8	8.0000000000000000E+00	1.1004915073699499E+00
y_2^L	1/2	5.0000000000000000E-01	9.0049150736994987E-01

Solution

$x_1^\star$ = 1.0000000000000000E+00
$x_2^\star$ = 0.0000000000000000E+00
$y_1^\star$ = 5.0000000000000000E-01
$y_2^\star$ = 1.0000000000000000E+00
$f_0^\star$ = -1.7500000000000000E+00

$u_1^\star$ = 4.0000000000000000E+00
$u_2^\star$ = 3.0000000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00
$u_4^\star$ = 0.0000000000000000E+00

Provenance

Floudas [15, §9.2.9] ← Bard [6, p90]. The location of the nonnegativity constraints is ambiguous in [15] but stated in [6]. In both [15] and [6] the outer constraint I call f_1 is an inner constraint given as $x_1 + x_2 \le 2$, but according to the author of this problem [5] it should be the outer constraint shown here; see also [16, p147].

Test Problem 26

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2$$

$$\text{subject to} \left\{ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y_1 - x_1)^2 + (y_2 - x_2)^2 \\ \text{subject to} \left\{ \begin{array}{l} g_1(\mathbf{y};\mathbf{x}) = -y_1 + \frac{1}{2} \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y_2 + \frac{1}{2} \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = y_1 - \frac{3}{2} \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = y_2 - \frac{3}{2} \le 0 \end{array} \right. \end{array} \right] \right.$$

Starting Bounds

	natural	fair	tight
x_1^H	$+\infty$	5.0000000000000000E+00	5.1042031701792523E-01
x_1^L	$-\infty$	4.9999999999999822E-02	4.9042031701792521E-01
x_2^H	$+\infty$	5.0000000000000000E+00	5.1014561653177515E-01
x_2^L	$-\infty$	4.9999999999999822D-02	4.9014561653177519D-01
y_1^H	3/2	1.5000000000000000E+00	5.1098691460679879E-01
y_1^L	1/2	5.0000000000000000E-01	4.9098691460679877E-01
y_2^H	3/2	1.5000000000000000E+00	5.1004915073699497E-01
y_2^L	1/2	5.0000000000000000E-01	4.9004915073699501E-01

Solution

$x_1^\star$	=	5.0000000000000000E-01	$u_1^\star$	=	0.0000000000000000E+00
$x_2^\star$	=	5.0000000000000000E-01	$u_2^\star$	=	0.0000000000000000E+00
$y_1^\star$	=	5.0000000000000000E-01	$u_3^\star$	=	0.0000000000000000E+00
$y_2^\star$	=	5.0000000000000000E-01	$u_4^\star$	=	0.0000000000000000E+00
$f_0^\star$	=	-1.0000000000000000E+00			

Provenance

Floudas [15, §9.3.7] ← Falk [12, p69] ⇜ de Silva [10, p58]. Falk and de Silva use y for the outer variable and x for the inner, so their notation is opposite to that used in Floudas and here. Falk [12] gives the source of this problem as de Silva [10], but the problem in [10] that most resembles this one (de Silva's Test Problem #1) has, after translating into our variable names, $f_0(\mathbf{x},\mathbf{y}) = x_1^2 - 3x_1 + x_2^2 - 3x_2 + y_1^2 + y_2^2$ so it is not the same as the problem given in [12].

Test Problem 27

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -x_1^2 y_1^2 + y_2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x_1^2 + x_2^2 - 1 \leq 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = x_1 y_1 + x_2 y_2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -y_1 \leq 0 \\ g_2(\mathbf{y};\mathbf{x}) = -y_2 \leq 0 \\ g_3(\mathbf{y};\mathbf{x}) = -x_1^2 - x_2^2 + y_1 + y_2 \leq 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x_1^H	1	1.0000000000000000E+00	-8.9579682982074793E-01
x_1^L	-1	-1.0000000000000000E+00	-1.0957968298207479E+00
x_2^H	1	1.2000000000000000E+00	1.0145616531775192E-01
x_2^L	-1	-1.0000000000000000E+00	-9.8543834682248088E-02
y_1^H	2	2.2000000000000002E+00	1.1098691460679877E+00
y_1^L	0	0.0000000000000000E+00	9.0986914606798763E-01
y_2^H	2	2.0000000000000000E+00	1.0049150736994984E-01
y_2^L	0	0.0000000000000000E+00	-9.9508492630050169E-02

Solution

$$y_1^\star(\mathbf{x}) = \begin{cases} 0 & x_1 \geq 0,\ x_2 \geq 0 \\ x_1^2 + x_2^2 & x_1 < 0,\ x_1 \leq x_2 \\ 0 & x_2 < 0,\ x_2 < x_1 \end{cases} \qquad y_2^\star(\mathbf{x}) = \begin{cases} 0 & x_1 \geq 0,\ x_2 \geq 0 \\ 0 & x_1 < 0,\ x_1 \leq x_2 \\ x_1^2 + x_2^2 & x_2 < 0,\ x_2 < x_1 \end{cases}$$

$x_1^\star$ = -1.0000000000000000E+00
$x_2^\star$ = 0.0000000000000000E+00
$y_1^\star$ = 1.0000000000000000E+00
$y_2^\star$ = 0.0000000000000000E+00

$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 1.0000000000000000E+00
$u_3^\star$ = 1.0000000000000000E+00
$f_0^\star$ = -1.0000000000000000E+00

There is also a local minimum at the origin.

Provenance
Dempe [9, p159]. Dempe uses $\mathbf{y}$ for the outer variables and $\mathbf{x}$ for the inner, so his notation is opposite that used here. In his notation he gives the optimal value function as $\varphi_0(\mathbf{y}) = -y_1^2(y_1^2 + y_2^2)$ for $y_1 < 0,\ y_1 \leq y_2$, but in our notation $f_0(\mathbf{x}, \mathbf{y}^\star(\mathbf{x})) = -x_1^2(x_1^2 + x_2^2)^2$. The outer problem given in [9] is unbounded, so I added the constraint $f_1(\mathbf{x},\mathbf{y}) = x_1^2 + x_2^2 - 1 \leq 0$.

Test Problem 28

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -x_1^2 - 3x_2 - 4y_1 + y_2^2$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x_1^2 + 2x_2 - 4 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x_1 \le 0 \\ f_3(\mathbf{x},\mathbf{y}) = -x_2 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = 2x_1^2 + y_1^2 - 5y_2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -x_2 - 3y_1 + 4y_2 + 4 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = -y_1 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y_2 \le 0 \end{cases} \end{array} \right] \end{cases}$$

Starting Bounds

	natural	fair	tight
x_1^H	2	2.4000000000000004E+00	1.0420317017925204E-01
x_1^L	0	-2.0000000000000000E+00	-9.5796829820747970E-02
x_2^H	2	4.4000000000000004E+00	2.1014561653177521E+00
x_2^L	0	0.0000000000000000E+00	1.9014561653177520E+00
y_1^H	26/5	7.3333333333333339E+00	1.9848691460679877E+00
y_1^L	2/3	6.6666666666666607E-01	1.7848691460679877E+00
y_2^H	17/5	3.3999999999999999E+00	9.1629915073699497E-01
y_2^L	0	-1.3999999999999999E+00	8.9629915073699495E-01

A starting point $\mathbf{x}^0 = [0,2]^\top$, $\mathbf{y}^0 = [4,1]^\top$ is given in [3], but that would make $x_1^0 = x_1^\star$ and $x_2^0 = x_2^\star$. Instead I selected bounds that make $\mathbf{x}^0 \neq \mathbf{x}^\star$.

Solution

$x_1^\star$ = 0.0000000000000000E+00
$x_2^\star$ = 2.0000000000000000E+00
$y_1^\star$ = 1.8750000000000000E+00
$y_2^\star$ = 9.0625000000000000E-01
$f_0^\star$ = -1.2678710937500000E+01

$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 1.2500000000000000E+00
$u_3^\star$ = 0.0000000000000000E+00
$u_4^\star$ = 0.0000000000000000E+00

Provenance

Bard [3, p24] $\leftrightsquigarrow$ Bard [2, p19]. In [2], where the problem is stated using maximizations, the first term in the outer objective is $2x_1^2$, but in [3], where the problem is stated using minimizations, the first term in the outer objective is $-x_1^2$ (as here). Thus the problems are similar but not the same. The location of the constraints I call f_1, f_2, and f_3 is ambiguous in [2] but specified in [3]. The algorithm described in [3] is said to find the global optimum, but the last iterate listed from it is $\mathbf{x}^1 = [1.45, 0.95]^\top$, $\mathbf{y}^1 = [1.88, 0.64]^\top$ which is slightly infeasible for both f_1 and g_1 and yields $f_0 = -12.0629$.

Test Problem 29

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x_1 + x_2 + y_1 - 2y_2 - 40 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x_1 \le 0 \\ f_3(\mathbf{x},\mathbf{y}) = x_1 - 50 \le 0 \\ f_4(\mathbf{x},\mathbf{y}) = -x_2 \le 0 \\ f_5(\mathbf{x},\mathbf{y}) = x_2 - 50 \le 0 \\ \mathbf{y} \text{ solves} \begin{bmatrix} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -x_1 + 2y_1 + 10 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = -x_2 + 2y_2 + 10 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = -y_1 - 10 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = y_1 - 20 \le 0 \\ g_5(\mathbf{y};\mathbf{x}) = -y_2 - 10 \le 0 \\ g_6(\mathbf{y};\mathbf{x}) = y_2 - 20 \le 0 \end{cases} \end{bmatrix} \end{cases}$$

Starting Bounds

	natural	fair	tight
x_1^H	50	5.0000000000000000E+01	1.0420317017925204E-01
x_1^L	0	0.0000000000000000E+00	-9.5796829820747970E-02
x_2^H	50	5.0000000000000000E+01	1.0145616531775192E-01
x_2^L	0	0.0000000000000000E+00	-9.8543834682248088D-02
y_1^H	20	2.0000000000000000E+01	-8.9013085393201230E+00
y_1^L	-10	-1.0000000000000000E+01	-1.0901308539320123E+01
y_2^H	20	2.0000000000000000E+01	-8.9950849263005015E+00
y_2^L	-10	-1.0000000000000000E+01	-1.0995084926300501E+01

Solution

$x_1^\star$ = 0.0000000000000000E+00
$x_2^\star$ = 0.0000000000000000E+00
$y_1^\star$ = -1.0000000000000000E+01
$y_2^\star$ = -1.0000000000000000E+01
$f_0^\star$ = 0.0000000000000000E+00

$u_1^\star$ = 0.0000000000000000E+00
$u_2^\star$ = 0.0000000000000000E+00
$u_3^\star$ = 2.0000000000000000E+01
$u_4^\star$ = 0.0000000000000000E+00
$u_5^\star$ = 2.0000000000000000E+01
$u_6^\star$ = 0.0000000000000000E+00

Provenance

Floudas [15, §9.3.4] ← Visweswaran [25, p160] ← Aiyoshi [1, p1114]. In [1] the optimal point is said to be $x_1 = 25, x_2 = 25, y_1 = 5, y_2 = 10$, but that point is infeasible for constraint g_2.

Test Problem 30

$$\min_{\mathbf{x},\mathbf{y}} f_0(\mathbf{x},\mathbf{y}) = -8x_1 - 4x_2 + 4y_1 - 40y_2 + 4y_3$$

$$\text{subject to} \begin{cases} f_1(\mathbf{x},\mathbf{y}) = x_1 + 2x_2 - y_3 - 1.3 \le 0 \\ f_2(\mathbf{x},\mathbf{y}) = -x_1 \le 0 \\ f_3(\mathbf{x},\mathbf{y}) = -x_2 \le 0 \\ \mathbf{y} \text{ solves} \left[\begin{array}{l} \min_{\mathbf{y}} g_0(\mathbf{y};\mathbf{x}) = 2y_1 + y_2 + 2y_3 \\ \text{subject to} \begin{cases} g_1(\mathbf{y};\mathbf{x}) = -y_1 + y_2 + y_3 - 1 \le 0 \\ g_2(\mathbf{y};\mathbf{x}) = 4x_1 - 2y_1 + 4y_2 - y_3 - 2 \le 0 \\ g_3(\mathbf{y};\mathbf{x}) = 4x_2 + 4y_1 - 2y_2 - y_3 - 2 \le 0 \\ g_4(\mathbf{y};\mathbf{x}) = -y_1 \le 0 \\ g_5(\mathbf{y};\mathbf{x}) = -y_2 \le 0 \\ g_6(\mathbf{y};\mathbf{x}) = -y_3 \le 0 \end{cases} \end{array} \right] \end{cases}$$

The upper-level constraint f_1 includes the lower-level variable y_3, so the inducible region of this problem might not be connected.

Starting Bounds

	natural	fair	tight
x_1^H	3/2	1.5000000000000000E+00	5.1014561653177515E-01
x_1^L	0	0.0000000000000000E+00	4.9014561653177519E-01
x_2^H	53/60	8.8333333333333333E-01	8.1074552025501290E-01
x_2^L	0	0.0000000000000000E+00	7.9074552025501288E-01
y_1^H	3/2	1.5000000000000000E+00	1.0986914606798769E-01
y_1^L	0	0.0000000000000000E+00	-9.0130853932012325E-02
y_2^H	3/2	1.5000000000000000E+00	2.1004915073699501E-01
y_2^L	0	0.0000000000000000E+00	1.9004915073699499E-01
y_3^H	2	2.0000000000000000E+00	8.1042031701792527E-01
y_3^L	0	0.0000000000000000E+00	7.9042031701792526E-01

Solution

$x_1^\star$	=	5.0000000000000000E-01	$u_1^\star$	=	0.0000000000000000E+00
$x_2^\star$	=	8.0000000000000000E-01	$u_2^\star$	=	5.0000000000000000E-01
$y_1^\star$	=	0.0000000000000000E+00	$u_3^\star$	=	1.5000000000000000E+00
$y_2^\star$	=	2.0000000000000000E-01	$u_4^\star$	=	7.0000000000000000E+00
$y_3^\star$	=	8.0000000000000000E-01	$u_5^\star$	=	0.0000000000000000E+00
$f_0^\star$	=	-1.2000000000000000E+01	$u_6^\star$	=	0.0000000000000000E+00

The multipliers are not uniquely determined, but $u_2 \ge \frac{1}{2}$ so I chose $u_2 = \frac{1}{2}$ and that determines the others.

Provenance

Luo [22, p357] ← Hansen [17, p1204]. In [22] the outer problem is stated as a maximization, and in [17] both problems are stated as maximizations. In [17] the correct solution is found but it is incorrectly reported as $\mathbf{x}^\star = [0.5, 0.8]^\top$, $\mathbf{y}^\star = [0.0, 2.0, 0.8]^\top$. That point is infeasible for g_1 and g_2.

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