Did Archimedes Do Calculus?

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Introduction

The works of Archimedes (c. 287–212 BCE) read like modern mathematics: fully-formed, to the point, and clever. Heath calls them "monuments of mathematical exposition" that are "so impressive in their perfection as to create a feeling akin to awe in the mind of the reader" [14]. Many histories of the calculus begin with Archimedes, and they highlight the parallels between his discoveries and results that are found today by limits, infinite series, and integration techniques, even going so far as to assert that the differences in approaches may be more in words than in ideas [4]. A student may wonder why, if Archimedes' discoveries are so novel and modern in style, he is not credited with the discovery of the calculus. Historians award that distinction unequivocally to Newton and Leibniz, who lived nearly two millennia after Archimedes. To investigate whether Archimedes should instead be acknowledged as the first mathematician to "do calculus," we will explore three aspects of his work that are related to the calculus: his mechanical method, which anticipates integration techniques; his quadrature of the parabola, which demonstrates how to handle infinite series; and his investigations into spirals, which involve properties of tangent lines.

As a whole, ancient Greek mathematicians banned infinity from their formal demonstrations, because at the time the concept was not grounded on any logical basis. Intuition often failed when considering questions of the infinite and how to divide the continuum, as exemplified by Zeno's famous paradoxes which, to some, have since been resolved by modern formulations of the calculus [4]. However, until the 19th century, most mathematicians were unwilling to accept infinity as anything more than a "potential," as exemplified by the "never-ending" infinite sequence $1, 2, 3, \dots$ versus the "completed" set $\{1, 2, 3, \dots\}$ [22]. How the ancient Greeks grappled with this idea is summarized in Aristotle's *Physics*:

> ...clearly there is a sense in which the infinite exists and another sense in which it does not ... magnitude is never actually infinite, but it is infinite by way of division—for it is not difficult to refute the theory of indivisible lines—the alternative that remains, therefore, is that the infinite exists potentially. [12]

In general, we find that the Greeks "stood still before the abyss of the infinite and never ventured to overstep the bounds of clear conceptions" [2]. For them, arguments relying on infinity simply would not pass logical muster, but concepts of infinity and its formal approaches, especially considering limits and continuity, are necessary components of the calculus.

Like many of his contemporaries, Archimedes possessed a "horror of the infinite" [9], but his methods were more refined than many of his peers' techniques. He proceeded cautiously and relied heavily on "sterile and rigorous" arguments by double *reductio ad absurdum*, especially in proofs involving area or volume [11]. That is, he used proofs by contradiction to show that an area can be neither less than nor greater than a given magnitude, so that the area in question must be equal to that magnitude. This indirect method of proof, whose discovery is credited to Eudoxus, is more appropriately called the *compression method* rather than the *exhaustion method*. As Dijksterhuis points out, the mode of reasoning arose from the idea that the infinite is inexhaustible, so the name "exhaustion method" is "about the worst name that could have been devised" [8]. The *approximation method* is another indirect method of proof, distinct from the compression method, in which Archimedes approximates a magnitude from below by a partial sum and then shows that the difference between the magnitude and partial sum can be made less than any given magnitude [8]. This treatment of infinite series closely resembles our current reasoning, and the only example of it in Archimedes' work occurs in his *Quadrature of the Parabola*, discussed below.

For easier understanding, we often present Archimedes' results in modern notation, but we do so at the risk of *mis*understanding the ancient Greeks' perspective. They had neither analytic geometry nor symbolic algebra, which, perhaps more than anything else, delayed progress toward the calculus as we know it today [6]. Greek mathematics lacked a general definition of number, and thus they did not develop any notion of variables representing continuous values [4]. Instead, their theory of proportion, as laid out in Book V of Euclid (and its discovery again credited to Eudoxus), involved geometric magnitudes as ratios to one another. For example, the statement, "the area of the circle is equal to πr^2," would be nonsense to Archimedes. Instead, he would tell us that the area of the circle is "the same as that of a right triangle with height equal to the radius of the circle and base given by the circumference of the circle" (as proven in *Measurement of a Circle*, Prop. 1). While it is interesting to note that Eudoxus' theory of proportion anticipated Dedekind cuts by over two millennia [22], the ancient Greeks did not perceive numbers, areas, or shapes in the way that we now perform calculations or plot curves. They were far more interested in geometry as an unchanging and ideal mode of reasoning, rather than a practical science, which would motivate the development of the calculus in the centuries to come.

Even though the Greeks formally did not speak of any kind of "completed sums" of infinite series or "infinitely close" approximations to limiting values, they likely arrived at these notions in their discovery processes. Writing in 1685, John Wallis hypothesized that "the Ancients had somewhat of a like nature with our Algebra" and that we should not think that "all these Propositions in Euclid and Archimedes were in the same way found out, as they are now delivered to us" [12]. Heath supports this view, supposing that the Greeks had techniques "hardly less powerful than those of modern analysis" and claiming that Archimedes' use of indefinitely narrow strips in *The Method* (shown below) "would be quite rigorous for us today, although it did not satisfy Archimedes himself" [2]. To Archimedes, his method of discovery merely indicated, but did not prove, that a result was correct. Because of this, he recast all his analysis in classical geometric constructions, much in the same way Newton would conceal his fluxions nearly 2,000 years later in *The Principia* [9]. Thus, Archimedes' readers are left to wonder: how could one man discover so much, and how could he craft such ingenious arguments to support his discoveries?

The Method: A Balancing Act

Archimedes' results are crown jewels in the Greek tradition of pure logic and precise argument, but it is clear that they were not discovered in the same way that they were proven. Until *On the Method of Mechanical Theorems, for Eratosthenes* (shortened to *The Method*) was discovered in 1906, the techniques that Archimedes used to find his results remained hidden. Literally hidden, in a palimpsest, partially erased underneath medieval prayers and gold-leaf illustrations, until the writing was fully analyzed in the 21st century using X-ray, infrared, ultraviolet light, and other advanced technology [17]. This manuscript (referred to as Codex C), copied sometime in the 10th century, is the only source we have for *The Method*, one of the most tantalizing documents in the history of mathematics.

In it, Archimedes writes to Eratosthenes (who was head librarian at Alexandria and who measured the circumference of the Earth) to explain how he found results that he previously communicated but did not prove. He encourages Eratosthenes to investigate the problems himself and suggests trying his "mechanical method" to get started, explaining:

> . . . certain things first became clear to me by a mechanical method, although they had to be demonstrated by a geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.[1]

Thus, the reader is provided an intimate glimpse into Archimedes' method of discovery, which reveals how he, long before he knew how to prove his theorems, became convinced of their truth [8].

His method revolves around the *law of the lever*, which, in a modern form, states that two objects are in equilibrium about a point (called the

[1]Unless otherwise noted, all translations of Archimedes' writings are from [2].

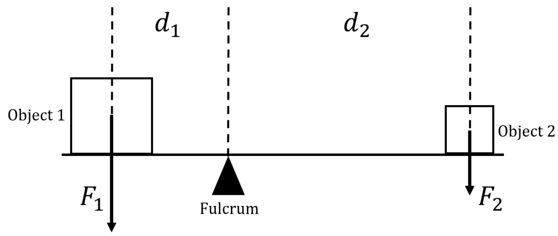

Figure 1: The law of the lever. The two objects balance if $F_1 d_1 = F_2 d_2$.

fulcrum) if their moments (or torque) about the point are equal. Referring to Figure 1, given two objects of different weight, F_1 and F_2, that are placed on a rigid beam at different distances, d_1 and d_2, from a fulcrum, they will balance if

$$F_1 d_1 = F_2 d_2 \tag{1}$$

However, Archimedes did not state the law like this. Starting with several postulates, in *On the Equilibrium of Planes* he writes the following in Propositions 6 and 7:

Theorem 1 (Archimedes' law of the lever) *Two magnitudes, whether commensurable [Prop. 6] or incommensurable [Prop. 7], balance at distances reciprocally proportional to the magnitudes.*

Algebraically, Archimedes restates Equation (1) as

$$\frac{F_1}{F_2} = \frac{d_2}{d_1} \tag{2}$$

We see that if two objects are the same density, we do not need to refer to their "weights" and can simply refer to their "magnitudes," whether we take that to mean mass, area, volume, or something else. Thus, Archimedes adapts a physical idea from mechanics to a more abstract mathematical technique. In this respect he is unique among the Greeks, who were predisposed to completely separating their mathematics from physical applications. Dijksterhuis claims that Archimedes was "the first to establish the close interrelation between mathematics and mechanics, which was to become of such far-reaching significance for physics as well as mathematics" [8].

To illustrate the method, let us find the volume of a sphere as he does in Proposition 2 of *The Method*. In modern notation, the volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$, and the volume of a right cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. If the radius and height of the cone are equal to the radius of the sphere, then the volume of the sphere is 4 times that of the cone. Here is how Archimedes expresses this:

Theorem 2 (Archimedes' volume of a sphere) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius.*

To see why this is true, let $ABCD$ be a great circle of a sphere, with $AC = BD$ as perpendicular diameters (Figure 2). In the same plane as circle $ABCD$, let the isosceles triangle AEF be a cone's cross section with height and radius equal to AC, and let rectangle $EFGL$ be a cylinder's cross section with height and radius equal to AC. That is, the triangle and rectangle both have heights equal to the diameter of the circle and have bases equal to twice the diameter of the circle. Extend AC to H so that $AH{=}AC$. Perpendicular to AC, construct MN connecting sides EL and FG. Referring to Figure 2, we wish to show that the volume of the sphere is equal to 4 times the volume of the cone ABD.

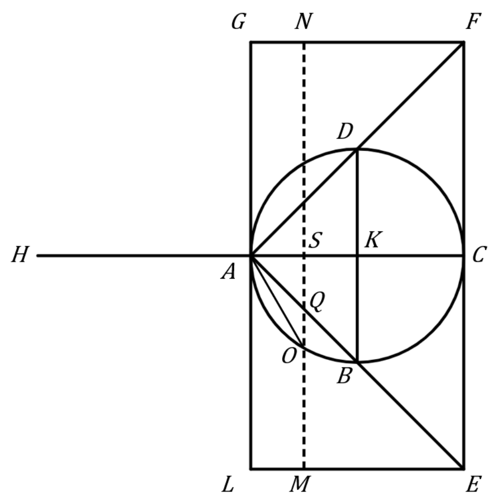

Figure 2: Volume of a sphere by Archimedes' mechanical method.

From Heath's proof [2], since $MS = AC = AH$ and $AS = QS$, we have

$$\begin{aligned} MS \cdot QS &= AC \cdot AS \\ &= AO^2 \\ &= OS^2 + QS^2 \end{aligned}$$

Now multiply $OS^2 + QS^2 = QS \cdot MS$ by πAH:

$$AH(\pi OS^2 + \pi QS^2) = QS \cdot \pi(AH \cdot MS)$$

$$AH(\pi OS^2 + \pi QS^2) = AS \cdot \pi MS^2 \tag{3}$$

Now we regard CH as a lever, with the fulcrum at A. Note that πOS^2 represents a circular cross section of the sphere, πQS^2 represents a circular cross section of the cone AEF, and πMS^2 represents a circular cross section of the cylinder. Applying the law of the lever, we see from Equation (3) that the sum of the cross sections of the sphere and cone at distance AH from the fulcrum will balance the cross section of the cylinder at distance AS from the fulcrum. As we change the placement of segment MN or "sweep" it across the rectangle, we obtain all circular cross sections of the solids. Taking all these cross sections and "hanging" them on either side of the fulcrum, we see that the sum of the sphere and cone AEF at distance AH balances the cylinder at distance AK (its center of gravity), or by Equation (2):

$$\frac{\text{cylinder}}{\text{sphere} + \text{cone } AEF} = \frac{AH}{AK}$$

But $AH = 2(AK)$, so the cylinder $= 2(\text{sphere}+\text{cone } AEF)$. Archimedes would have known from Euclid (XII.10) (and he states so in the introduction of *On the Sphere and Cylinder I*) that the cone AEF has one third of the volume of the cylinder in which it is inscribed. Therefore, cone $AEF =$ 2(sphere), and by construction, $EF = 2BD$, so

$$\begin{aligned} \text{cone } AEF &= 2^3(\text{cone } ABD) \\ 2(\text{sphere}) &= 8(\text{cone } ABD) \\ \text{sphere} &= 4(\text{cone } ABD) \end{aligned}$$

which was to be shown.

Of course, Archimedes did not demonstrate his proposition exactly like this (especially the step where we multiply by π), but the fundamental idea of his method is still present: to find an area or volume, cut it up into a very large number of thin parallel strips and hang the pieces on the end of a lever so that they balance with a known shape. In doing so, Archimedes regards surfaces as being made up of lines and in turn, solids made up of surfaces. To most students first learning integral calculus, this idea would seem very familiar and intuitive. In his introduction to Archimedes' works, Heath even refers to the procedure as "genuine integration" [2].

Although Archimedes captured the spirit of an integral, to claim that Archimedes performed integration is to misinterpret the strict definition of an integral, as the limit of an infinite series and not as the sum of an infinite number of points, lines, or surfaces [4]. Archimedes' method is only "rigorous enough for us today" if we grant him our modern definitions of number, limit, and continuity. Furthermore, Sarton states that it is misleading to even use the word "method," because Archimedes did not have a general way to compute integrals, and he calls each solution "rigorous but inapplicable to other problems" [19]. Thus, rather than speaking of Archimedes performing integration, it is more correct to say that Archimedes *anticipated* integration, or that, of the ancients, he came the *nearest* to actual integration.

After demonstrating the volume of the sphere in *The Method*, Archimedes goes on to conclude that a sphere inscribed in a cylinder has a volume in a 2:3 ratio with that of the cylinder. This is formally proved with his compression method and stated as a corollary after Proposition 34 in *On the Sphere and Cylinder I*, a work believed to have been written just after *The Method* [14]. Furthermore, he proved that the same sphere and cylinder also have surface areas in a 2:3 ratio. Archimedes so highly regarded this beautiful result that according to Plutarch he wished the shapes to be engraved on his tombstone [18]. When Cicero served as quaestor in Syracuse in the 1st century BC, he searched the "great many tombs at the gate Achradinae" and found "a small column standing out a little above the briers, with the figure of a sphere and a cylinder upon it" [7]. Archimedes' tomb has since been lost, and we can only hope that, like *The Method*, it may one day be discovered again.

Sum Discoveries with Parabolas

Another one of Archimedes' most celebrated discoveries is his determination of the area of a segment of a parabola, or what he would have called an *orthotome*, a section of a right-angled cone [8]. While mourning the loss of his friend Conon, Archimedes writes to Dositheus in *Quadrature of the Parabola* that he did not believe any of his predecessors had attempted the problem and that the area was "first discovered by means of mechanics and then exhibited by means of geometry" [2]. A demonstration by the mechanical method takes up roughly the first half of *Quadrature of the Parabola*, while the latter part is devoted to a formal proof using his approximation method with a geometric series.

Witnessing the power of his mechanical method, it may seem strange that Archimedes would seek an alternative demonstration for a formal proof. According to Dijksterhuis, "when Archimedes denies the demonstrative force of his mechanical method which he explains to Eratosthenes, he does not do so on account of its mechanical nature, but exclusively because it makes use of the method of indivisibles" [8]. It is unclear how exactly Archimedes meant for his cross sections to be to be understood, either as being "infinitely thin" or having a "very small but non-zero" width. Because he viewed them physically balancing like thin strips or laminae, he likely held beliefs similar to Democritus' atomic theory, where there exist smallest indivisible bodies from which everything is composed [3]. Eves supports this, claiming that the mechanical method has "the fertility of the loosely founded idea of regarding magnitude as composed of a large number of atomic pieces" [11]. Nevertheless, Archimedes realized he could not justify the use of indivisibles with the mathematical tools at his disposal, and for this reason he sought geometric proofs to his findings. Plutarch's 1st century account captures Archimedes' attitude toward his mechanical discovery process vs. proof with pure geometry:

> Regarding the business of mechanics and every utilitarian art as ignoble and vulgar, he gave his zealous devotion only to those subjects whose elegance and subtlety are untrammelled by the necessities of life... in them the subject-matter vies with the demonstration, the former possessing strength and beauty, the latter precision and surpassing power... [18]

Of course, the area of a parabolic segment can be found today with integral calculus, and it is one of the first results a student will learn. Although the computation is easy with the machinery of modern formulas, it should be reiterated that Archimedes lacked analytic geometry and symbolic algebra. Thus, his techniques are not merely routine algorithms, but intricate arguments that utilize double *reductio ad absurdum*, special properties of conic sections, and even notions of infinite geometric series. Like his work with the volume of a sphere, his goal was to compare the area of the parabolic segment to a more well-known shape, in this case a triangle:

Theorem 3 (Archimedes' area of a parabolic segment) *Every segment bounded by a parabola and a chord is equal to four-thirds of the triangle which has the same base as the segment and equal height.*

We will now explore how Archimedes proved Theorem 3 with his approximation method, as in Proposition 24 of *Quadrature of the Parabola*. Let ABC be a parabolic segment which is bounded by the chord AC, and let D be the midpoint of AC (Figure 3). From the definitions given after Proposition 17, Archimedes defines the "base" of the parabolic segment as AC and the "vertex" of the segment as B, the point from which the greatest perpendicular to base AC is drawn. (Another way to describe the vertex B is that it is the point at which the tangent to the parabola will be parallel to the base AC [20, 21].) Note that Archimedes' definition in this context is different from our modern definition of the vertex of a parabola.

Construct triangle ABC, which is inscribed in the parabolic segment. Referring to Figure 3, we wish to show that the area of the parabolic segment is $\frac{4}{3}$ the area of triangle ABC.

Next, define a smaller parabolic segment APB with AB as its base and P as its vertex. Construct triangle APB as above, so that it is inscribed between the parabola and triangle ABC. Similarly, define a parabolic segment BQC, where BC is its base and Q is its vertex, and construct triangle BQC. This construction can be continued, inscribing smaller and smaller triangles by the same procedure. Thus, we construct a many-sided polygon inscribed in segment ABC that approximates the area of the segment from below.

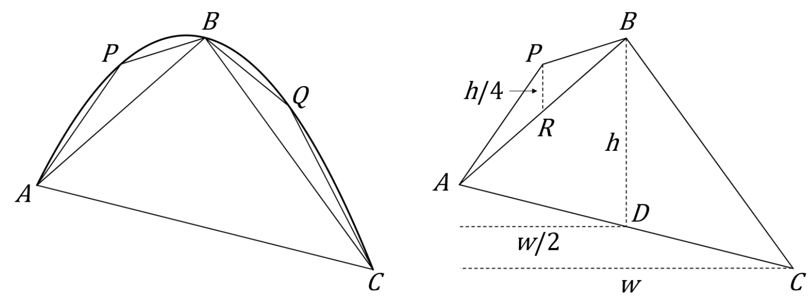

Figure 3: Area of parabola by Archimedes' approximation method.

In Propostion 21, Archimedes establishes that the height PR is $\frac{1}{4}$ of BD, and the width of triangle APB is $\frac{1}{2}$ that of ABC (Figure 3) [21]. So, triangles APB and BQC are each $\frac{1}{8}$ of the area of ABC. Together, they are $\frac{1}{4}$ of the area of ABC. As we construct more and more triangles, the two new inscribed triangles in each segment will bear the same 1:4 ratio that APB and BQC bear to ABC.

At this point in the proof we must digress to explain how these triangular pieces can now be added up. Let the area of triangle ABC be A. From a modern understanding, if we perform our construction n times, the approximate area of the parabolic segment is given by the partial sum S_n with a_n summands:

$$S_n = A\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}} + \frac{1}{4^n}\right)$$

Letting $n \to \infty$, the sum of the geometric series approaches $\frac{4}{3}$. Therefore, the area of the parabolic segment is $\frac{4}{3}A$, which was to be shown.

However, passing to a limit is precisely what Archimedes did *not* do [5]. We again find in Aristotle's *Physics* a summary of the Greek view on infinite series:

> ...as we see the magnitude being divided *ad infinitum*, so, in the same way, the sum of successive fractions when added to one another will be found to tend towards a determinate limit. For if, in a finite magnitude, you take a determinate fraction of it and then add to that fraction in the same ratio, and so on, but *not* each time including one and the same amount of the original whole, you will not traverse [i.e. exhaust] the finite magnitude. (Clarification by [12])

Instead of "exhausting" an infinite sum of triangles, Archimedes considers the remainder between the parabola and the inscribed many-sided polygon. In Proposition 23, he proves the following (using Heath's notation):

Theorem 4 (Archimedes' geometric sum) *Given a series of areas A, B, $C, D, \ldots Z$, of which A is the greatest, and each is equal to four times the next in order, then*

$$A + B + C + \cdots + Z + \frac{1}{3}Z = \frac{4}{3}A$$

If for simplicity we let $A = 1$, the equation above is equivalent to

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}} + \frac{1}{4^n} + \frac{1}{3}\left(\frac{1}{4^n}\right) = \frac{4}{3} \tag{4}$$

To see why the equation holds true, we will use modern notation, but let us not forget that Archimedes explained his reasoning in words, not symbols. Note that

$$\frac{1}{4^n} + \frac{1}{3}\left(\frac{1}{4^n}\right) = \frac{4}{3}\left(\frac{1}{4^n}\right) = \frac{1}{3}\left(\frac{1}{4^{n-1}}\right)$$

Thus, the left side of Equation (4) becomes

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-2}} + \frac{1}{4^{n-1}} + \frac{1}{3}\left(\frac{1}{4^{n-1}}\right)$$

That is, we replace our last two terms by the term $(1/3)(1/4^{n-1})$, which we will refer to as the "remainder" left between the parabola and the approximating polygon. In the process above, we are effectively lowering our largest exponent, n, to $n-1$, and as we repeat this process, our remainder can be made less than any assigned magnitude [8, 21]. In a finite amount of steps, the left side of Equation (4) will telescope to $1 + \frac{1}{3}(1)$, or $\frac{4}{3}$.

Many of us would conclude our proof here, but Archimedes kept going. He proceeds to use Theorem 4 and double *reductio ad absurdum* to show that the area of the polygon inscribed in the parabola can be neither less than nor greater than $\frac{4}{3}$ triangle ABC, so the area of the parabolic segment must be equal to $\frac{4}{3}$ triangle ABC.

First, suppose the area of the polygon is greater than $\frac{4}{3}$ triangle ABC. By Theorem 4, if we let the area of triangle ABC be A, we see that $A + B + C + \cdots + Z < \frac{4}{3}A$, contradicting our supposition.

The second contradiction argument is much more subtle (see [8] for more details). Suppose the area of the polygon is less than $\frac{4}{3}$ triangle ABC. Let Σ denote the area of the parabolic segment, let $K = \frac{4}{3}$(triangle ABC)$= \frac{4}{3}A$, and let S_n denote the area of the polygon after performing our construction n times, where a_n is its final summand. Referring to Theorem 4 and Equation (4), we find n so that

$$\frac{1}{4^n}A < \frac{4}{3}A - \Sigma \qquad \text{or} \qquad a_n < K - \Sigma$$

In other words, we perform our construction enough times to arrive at an area that is less than the difference between $\frac{4}{3}$ triangle ABC and the area of the segment. We see that K now exceeds S_n by an area less than a_n, and Σ exceeds S_n by an area greater than a_n, or

$$K - S_n < a_n < K - \Sigma$$

Therefore, $S_n > \Sigma$, which contradicts our supposition. Thus, since the area of the parabolic segment is neither greater than nor less than $\frac{4}{3}$ triangle ABC, it must be equal to $\frac{4}{3}$ triangle ABC, which was to be shown.

In Archimedes' approximation method we see one of the most sophisticated uses of a double *reductio ad absurdum* in ancient times. His treatment of an infinite geometric series with finite partial sums is remarkably similar to how Cauchy and others in the 19th century would handle the process, by producing a target value and proving that the series cannot be either greater or less than that value. To a modern mathematician, an infinite series is the succession of approximations by finite sums, and the Archimedean understanding has become a foundational part of real analysis [5].

The series above is not the only one found in Archimedes' work. In *On Conoids and Spheroids*, he goes beyond the two-dimensional area of a parabolic segment and finds the volume of a segment of a paraboloid. This result was so far ahead of its time that it was not rediscovered for another millennium, when Middle Eastern mathematicians independently proved it (though they had access to many of Archimedes' works, to our knowledge they did not possess *On Conoids and Spheroids* [15]). By slicing the solid into "equal parts by planes parallel to the base," his work with conoids may be the closest Archimedes came to actual integration. The series, which is only one aspect of the proof, and its geometrical demonstration are "in broad outline equivalent to performing the integration indicated by $\int_0^a x dx$" [4]. In addition, while calculating the area of a turn of his spiral in *On Spirals*, he supplies a series that is equivalent in polar coordinates to $\int_a^b \theta^2 d\theta$ [9]. This is another instance of his phenomenal ingenuity, but it is also another case where his technique is specific to the problem at hand and not applicable generally.

To say that Archimedes "computed" $\int x dx$ or $\int x^2 dx$ is to impute a modern bias and misunderstand Archimedes' intentions. By all indications, it seems that he wished to construct the solution to interesting geometric problems and not to generalize his results into any kind of new branch of mathematics [15]. This is in contrast to the methods of the handful of mathematicians in the 17th century who found demonstrations of $\int x^k dx$ for higher powers of x which led directly to algorithms of the calculus [4]. Furthermore, Archimedes' use of a geometric series in the quadrature of a parabola appears to be unique to quadratic functions; his method does not yield geometric series in general for other segments of plane curves [23]. Thus, it is doubtful whether Archimedes should be credited with discovering any general integration formulas or processes. However, this should not belittle what he did discover. Instead, our discussion highlights just how far ahead his ideas were, by preceding integral calculus and the rigorous treatment of infinite series by two millennia. We regard his determination of the area of a parabolic segment as important not so much because it helps us to compute areas, but because it suggests a way to define the general concepts of area and the integral [1].

Touching on Spirals

So far we have addressed problems concerning area, volume, and accumulation, which historically precede problems about tangents, slopes, and rates of change. These two problem types and their inverse relationship constitute the fundamental theorem of the calculus. As nearly all ancient Greek mathematicians were also philosophers, they stressed the abstract, ideal state of things, because "[r]elationships in the material world were subject to change and hence did not represent ultimate truth, but relationships in the ideal world were unchanging and absolute truths" [16]. Thus, they were more concerned with form than variation, and results related to differential calculus only exist in a few isolated cases [4]. In general, understanding the derivative as an instantaneous rate of change or as the slope of a tangent line is more intuitive when it is in the context of a functional relationship, which would not really come about until after the innovations of algebra and analytic geometry. Specifically, tangent line constructions would not become a widespread topic of investigation until about 1635, beginning with the work of Fermat. However, studies of motion were apparent in astronomical works, including those of Hipparchus of Nicaea, who lived about a century after Archimedes. He gave the first functional relationship between the chord and the arc of a circle, and over many centuries this would evolve into the sine function [6].

At the beginning of Book III of his *Elements*, Euclid gives the definition, "A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle" [10]. We see that his definition of a tangent line relies on the imprecise word "cut" (meaning that a line intersects the circle more than once, or divides it into two parts), and it is restricted only to that of a circle. Furthermore, the property that the tangent line will only intersect the circle once does not hold in general for other curves (such as an Archimedean spiral with multiple turns, see below). In Proposition 16 of the same book, Euclid refers to a line tangent to the circle such that "into the space between the straight line and the circumference another straight line cannot be interposed." This definition was adopted by Apollonius, who generalized tangent lines to other conic sections [8]. Virtually no methods

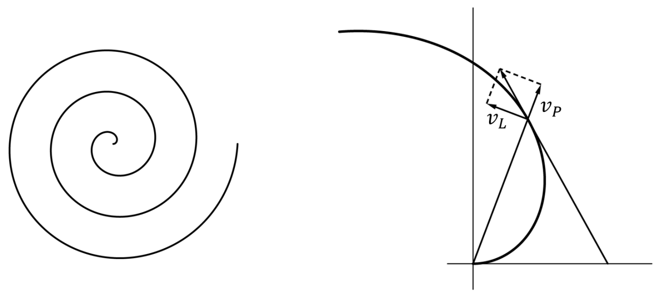

Figure 4: Archimedean spiral with 3 turns and Boyer's interpretation of the tangent line as the resultant of two-fold motion.

for the construction of tangent lines exist in Greek mathematics, apart from Apollonius' work and an isolated example in Archimedes' construction of a tangent to his spiral [9]. Archimedes' result is found in the aptly named *On Spirals*, arguably his most beautiful work.

Archimedes defines his spiral after Proposition 11:

> If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral in the plane.

Thus, the Archimedean spiral is generated by two uniform motions: by the line rotating about the origin of the spiral, and a point moving outward from the origin along the line. By construction, as Archimedes states in Proposition 12, if lines are drawn from the origin at equal angles between each other, then the points of intersection will be separated by a constant distance, i.e. they will be in arithmetic progression. Similarly, any line drawn from the origin intersects successive turnings of the spiral in points with a constant distance, giving a spiral with multiple turns the appearance of a "constant width" between turns (Figure 4).

As the Greeks were more interested in stationary forms, it is noteworthy that Archimedes describes his spiral in terms of moving bodies. Though Archimedes did not express the idea, Boyer hypothesizes that he borrowed ideas from kinematics in the same way that he drew from mechanics in *The Method* [4]. Archimedes may have had some notion of the two-fold uniform motion acting like vectors: the generating line with uniform velocity v_L rotates perpendicular to the point moving away from the origin at uniform velocity v_P, and the resultant, given by the parallelogram rule, produces the tangent line to the spiral (Figure 4). Although he would surely understand

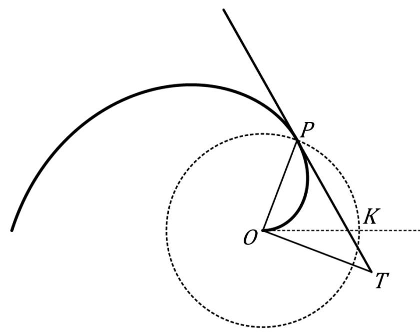

Figure 5: The tangent line PT can be drawn such that $OT =$ (arc KP).

the idea, we do not possess hard evidence that Archimedes discovered this on his own.

Unfortunately, Archimedes did not add much to the Euclidean definition of a tangent line, but he did extend the definition to his spiral and provides a unique way to construct the tangent line in Proposition 20:

Theorem 5 (Archimedes' tangent line to spiral) *If P be any point on the first turn of the spiral and OT be drawn perpendicular to OP, OT will meet the tangent at P to the spiral in some point T; and, if the circle drawn with center O and radius OP meet the initial line in K, then OT is equal to the arc of this circle between K and P measured in the "forward" direction of the spiral.*

This result is illustrated in Figure 5, which shows a spiral generated by the initial line through KO rotating counter-clockwise. Essentially, OT is drawn perpendicular to OP and equal in length to arc KP. The tangent line to the spiral at point P is found by joining PT. Of course, constructing a straight line with length equal to the arc of a circle is in general not possible by classic compass and straight-edge methods, because the length may involve a multiple of π, which is transcendental and therefore not constructible. (Recall that every constructible number is the root of some polynomial equation with rational coefficients [13].) Thus, the construction of the tangent line is assumed in Proposition 20.

Archimedes proves Theorem 5 by his signature double *reductio ad absurdum*, and we will give an outline below (for full details see [2]). In his argument, he makes use of constructions by *neusis* ("insertion" or "inclination"), where a straight line has to be drawn through a given a point and from which two given curves cut off a segment of given length [8]. Archimedes' proofs for each *neusis* that we will need below are contained in Propositions 7 and 8 of *On Spirals*. Though some have pointed out the logical gaps in Archimedes' reasoning [8, 14], we will simply assume the validity of the *neu-*

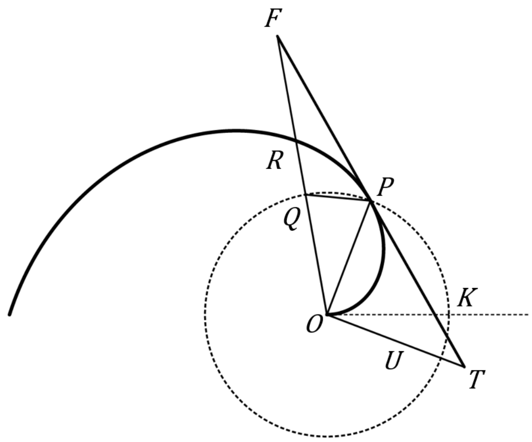

Figure 6: Archimedes' construction of his first *neusis*.

sis constructions. Referring to Figures 6 and 7, we wish to show that OT is equal to the length of the arc KP.

First, suppose OT is greater than the arc KP. We draw OU such that (arc KP) $< OU < OT$ (Figure 6). By Proposition 7 (*neusis*), it is possible to draw OQF such that

$$\frac{FQ}{PQ} = \frac{OP}{OU}$$

Then

$$\frac{FQ}{OQ} = \frac{PQ}{OU} < \frac{\text{arc } PQ}{\text{arc } KP} \quad \text{and} \quad \frac{OF}{OQ} < \frac{\text{arc } KQ}{\text{arc } KP} < \frac{OR}{OP}$$

But $OQ = OP$, so $OF < OR$, which is impossible. Therefore, OT is not greater than the arc KP.

Next, suppose OT is less than the arc KP. We draw OU' such that $OU' > OT$ but $OU' <$ (arc KP) (Figure 7). By Proposition 8 (*neusis*), it is possible to draw OKF' such that

$$\frac{F'Q'}{PG} = \frac{OP}{OU'}$$

Then

$$\frac{F'Q'}{OQ'} = \frac{PG}{OU'} > \frac{\text{arc } PQ'}{\text{arc } KP} \quad \text{and} \quad \frac{OF'}{OQ'} < \frac{\text{arc } KQ'}{\text{arc } KP} < \frac{OR'}{OP}$$

But $OQ' = OP$, so $OF' < OR'$, which is impossible. Therefore, OT is not less than the arc KP.

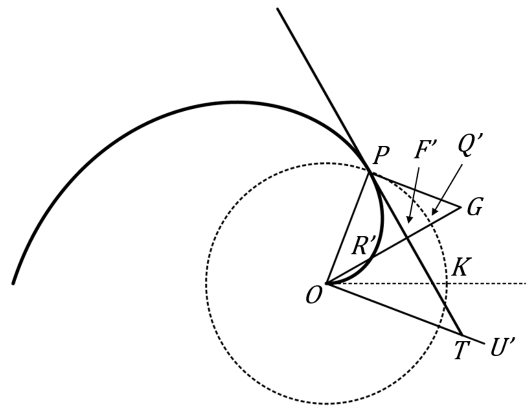

Figure 7: Archimedes' construction of his second *neusis*.

Since OT is neither greater than nor less than arc KP, it must equal KP, which was to be shown.

Overall, Archimedes' treatment of tangent lines is far from the ideas of differential calculus. His viewpoint is static and does not involve any ratios of change, which is unsurprising given that the notion of tangents and slopes as rates of change is reliant upon concepts of functions and limits. Nowhere in Greek mathematics is there the recognition of the need for limits, either for determining areas or tangents, and "even for the very definition of these ideas which intuition vaguely suggests" [4]. It is especially for his treatment of tangents that we cannot ascribe to Archimedes the discovery of calculus: he did not see the relationship between integration and differentiation which is fundamental to the calculus as a full-fledged branch of mathematics. This feat was not accomplished until the 17th century, when the time was ripe for discovery, and Newton and Leibniz seized upon ideas which had been accumulating for millennia.

Conclusion

We find in Archimedes' astounding list of discoveries many topics that can now be tamed with methods of the calculus: areas of circles, spirals, parabolas, and other conic sections; volumes of spheres, cylinders, cones, paraboloids, and ellipsoids; surface areas of spheres and spheroids; summations of geometric series; approximations for π and $\sqrt{3}$; tangent lines to curves; and applications in mechanics, hydrostatics, and centers of gravity. Our investigations above only constitute a small fraction of his pioneering work which preceded the development of the calculus. However, the fact that Archimedes addressed many problems that now appear in today's calculus classrooms does not imply that he necessarily "did" calculus.

Ultimately, the delineation between calculus and *the* calculus as a fully realized branch of mathematics involves a recognition of the inverse relationship between area problems (integrals) and tangent problems (derivatives), i.e. the fundamental theorem of the calculus. Many historians also look for general methods or computational algorithms for tackling related problems, instead of them being treated on a case-by-case basis. These criteria are not met in the works of Archimedes. In general, he consistently exploits special properties of geometric constructions, and he tends not to take advantage of previous solutions to similar problems [9]. He does not explicitly introduce a limit concept, although most modern analyses of his work (including the one presently) make note of equivalent results using passages to the limit and limit-definitions of integrals. We have seen that he does not perform integration, but rather, he uses clever devices for *avoiding* integration.

We might say that Archimedes anticipated the calculus in that he had many of the pieces, but the missing pieces and the tools to complete the puzzle were not available until the 17th century. The conclusion that he did not do calculus does not nullify his outstanding achievements, which spanned the entirety of mathematics known during his lifetime. He expanded the field to include ideas so profound that they were not improved upon for centuries to come. Of his mechanical method, he imagined that "some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me" [2]. Thus, Archimedes highlights the importance of understanding the techniques of our predecessors: that we might, by imitation or by extension, discover today what they could not.

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