Mathematics in Art and Architecture GEK1518

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Polygons and Tilings

Polygons

Polygons have *sides* and *corners.* It is called regular if all the sides and angles are equal. The vertex angle in a regular n-gon is 180 (n-2)/n. To see this, divide the polygon into n triangles. Below are the Greek names for some numbers. Instead of trigon we say triangle, and instead of tetragon we say quadrilateral. For n>4, I will often assume that a polygon is regular, unless stated otherwise.

| 3	4	5	6	7	8	9	10	11	12	20
Tri	Tetra	Penta	Hexa	Hepta	Octa	Ennea	Deca	Hendeca	Dodeca	Icosa
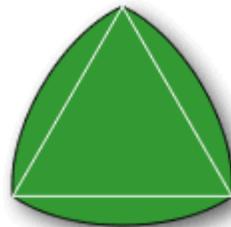
Some coins are polygonal, but if you look closely, you will see that the sides are not exactly straight. In fact, they are arcs of circles with centers at the opposite corner, as in the picture to the right. It is called a Reuleaux triangle, named after the German engineer Franz Reuleaux (1829-1905). It has the property that it has constant width. Do you see why? All the points on the arc have the same distance from the opposite corner. You can construct other examples of constant width curves by starting with a polygon with an *odd* number of sides and rounding them like above. Try to convince yourself why you cannot make a polygon with an even number of sides into a curve with constant width. Why do we want coins to have constant width? It makes it easier for vending machines to recognize them.

Tilings

Tilings or tessellations are ways of filling up the plane with tiles. They are often used when making floors. We will assume that our tilings satisfy the following:

1) The tiles are regular polygons.
2) The tiling is edge-to-edge. This means that two tiles intersect along a common edge, only at a common vertex or not at all.
3) All the vertices are of the same type. This means that the same types of polygons meet in the same order (ignoring orientation) at each vertex.

The left picture below is a tiling that is not edge-to-edge ([1]). An edge of a hexagon coincides with only half the edge of a triangular tile. The right picture is a tiling with more than one type of vertex, and where also the tiles are not regular polygons.
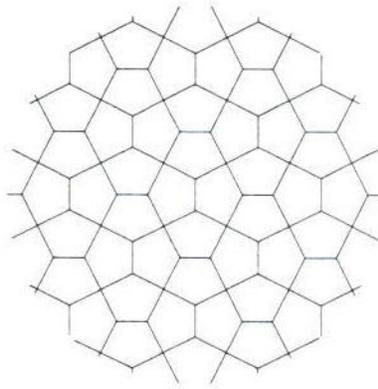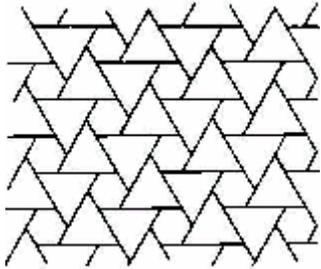

A vertex is of *type* $n_1.n_2\ldots n_r$ if it is surrounded in cyclic order by regular n-gons of order $n_1, n_2,\ldots, n_r$. We will denote 3.3.3.4.4 as $3^3.4^2$, and so on. When referring to the tiling, we will write $(3^3.4^2)$. It is possible to study more general tilings, where the vertices are not all of the same type, and in that case we need to have separate notation for tilings and vertices. In these notes we will ignore this.

There are only three tilings that use only one type of tiles. We will call them *Platonic* or regular tilings. There must be at least three polygons at each vertex. (Why?) There cannot be more than six. (Why?) There cannot be five. (Why?)
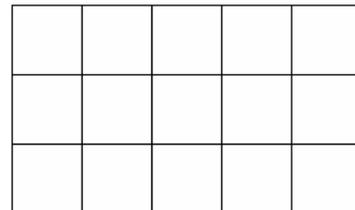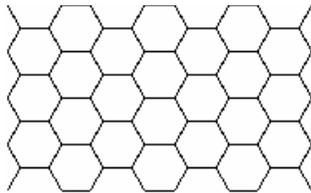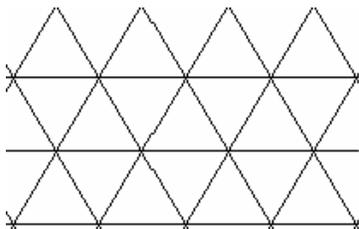

There are eight tilings that use more than one type of tiles. They are called *Archimedean* or semiregular tilings. The picture ([1]) shows the three Platonic and eight Archimedean tilings (notice that (3.4.6) occurs in two forms)

(3^6), (4^4), (6^3), $(3^4.6)$, $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), $(4.6.12)$, (4.8^2).

(3^6) (4^4) (6^3) $(3^4.6)$

$(3^4.6)$ $(3^3.4^2)$ $(3^2.4.3.4)$ $(3.4.6.4)$

$(3.6.3.6)$ (3.12^2) $(4.6.12)$ (4.8^2)
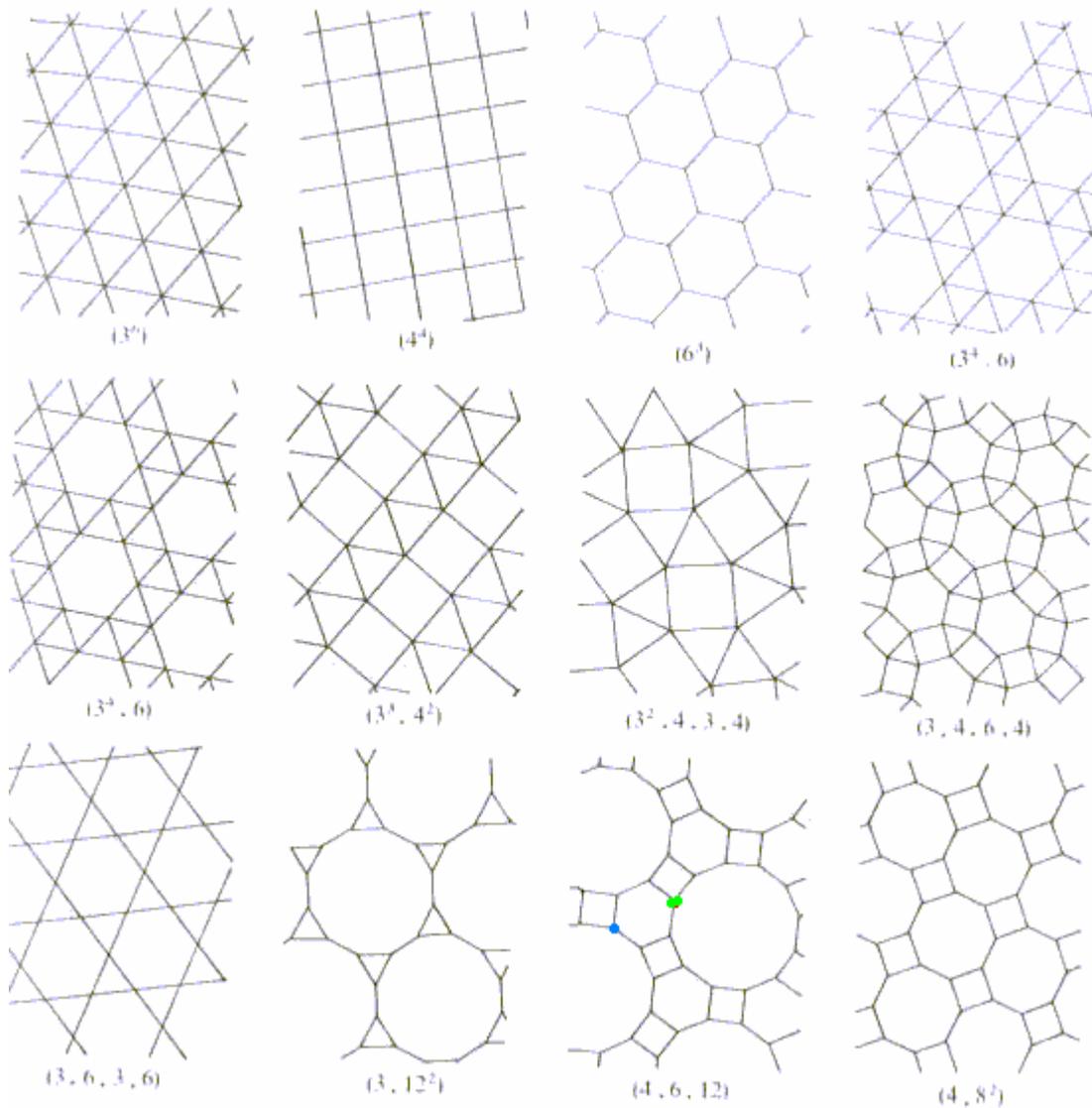
The vertex type 4.6.12 is special, in that it has two different orientations. Labeling a vertex in the clockwise direction results in a different notation from labeling the vertex in the anti-clockwise direction. For the other tilings, direction makes no difference in the notation. In the clockwise direction, the blue vertex is 4.6.12 while the green one is 4.12.6. Notice that we need to use both types in order to tile the plane.

Notice hat $3^4.6$ has two kinds of tilings that are congruent to each other by a reflection. That is, one is an image of the other under a reflection. We say that these two tilings are *enantiomorphic*.

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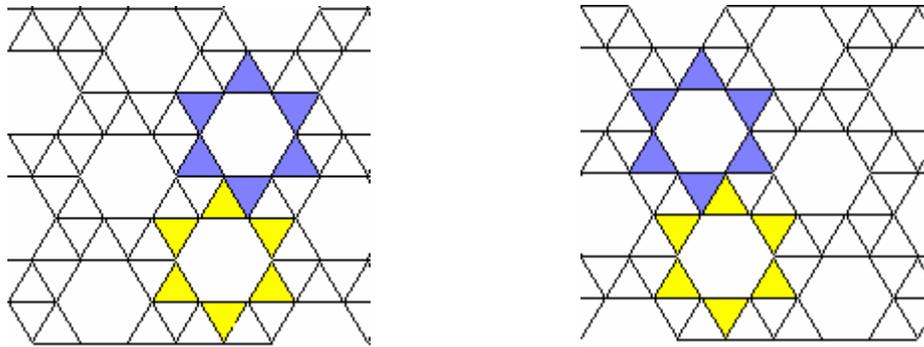
In the left picture, the topmost yellow triangle slides to the left of the bottommost blue triangle. In the right picture, the topmost yellow triangle slides to the right of the bottommost blue triangle.

Look carefully at the following tilings by vertex type $3^2.4.3.4$. If you look at the upright squares with triangles around them, it looks like the left one is a left-handed version while the right one is a right-handed version, just like the example above (http://www.uwgb.edu/dutchs/symmetry/archtil.htm).
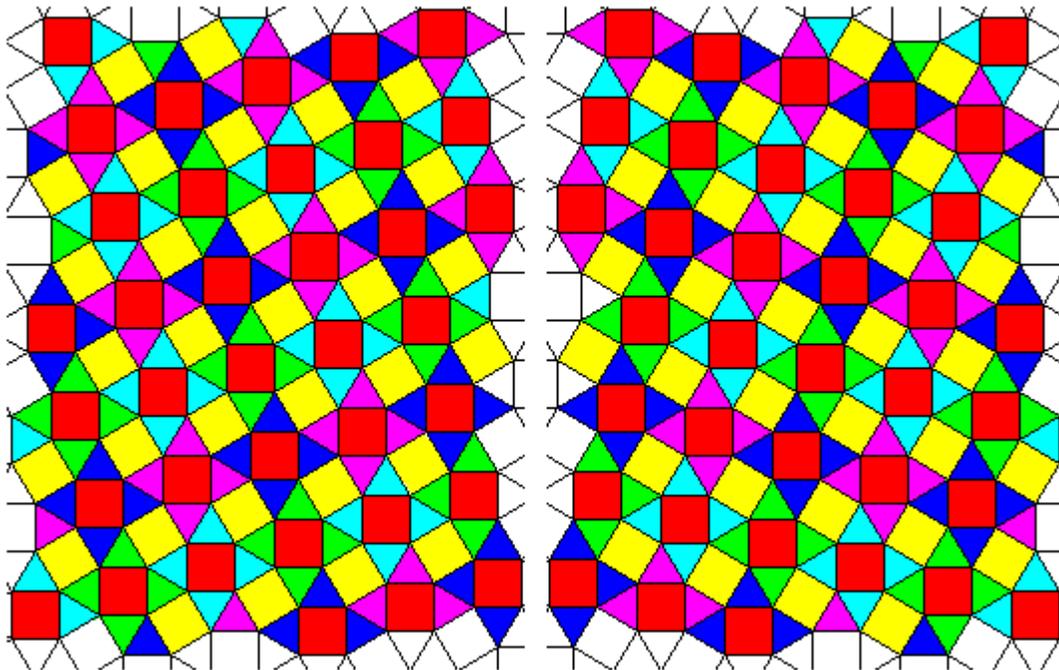
In fact, the above two tilings are not enantiomorphic. If you rotate the left picture to the right, you will get the picture below, and if you rotate further, you will get the right picture. In the left picture, the red squares are "right-handed", while the yellow squares are "left-handed". So you see that the pattern is a combination of left and right squares, and by lining one of families straight up, you focus on them. In the tiling below, you can see that there is a reflection axis. The reason why $3^4.6$ has in two forms, is that neither has a reflection axis.
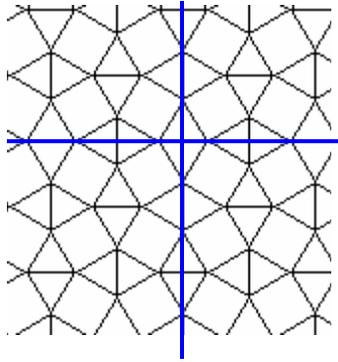
The number of each kind of regular polygon surrounding a vertex is called the *species* of the vertex. Notice that $(3^3.4^2)$ and $(3^2.4.3.4)$ have the same species but different types.

$(3^3.4^2)$

$(3^2.4.3.4)$
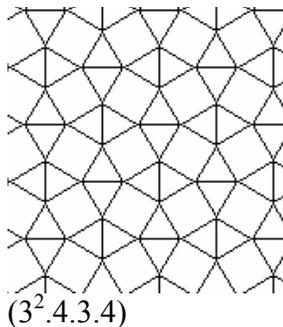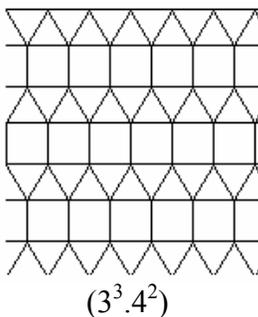
Why are these 11 tilings the only possible? I will not give a complete proof, but I will give an outline. Let us first determine what are the possible species of a vertex. The vertex angle of a regular n-gon is given by $180(n-2)/n$. Since the total number of degrees at a vertex is 360°, we get the following equation:

$$360^o = \Sigma\ 180(n_i-2)/n_i$$

Solving the above equation, we get 17 solutions. These are the species of vertices, and give rise to 21 types of vertices.

4) 3.3.3.3.3.3
5) 3.3.3.3.6
6) i) 3.3.3.4.4 and ii) 3.3.4.3.4
7) i) 3.3.4.12 and ii) 3.4.3.12
8) i) 3.3.6.6 and ii) 3.6.3.6
9) i) 3.4.4.6 and ii) 3.4.6.4
10) 4.4.4.4
11) 3.7.42
12) 3.8.24
13) 3.9.18
14) 3.10.15
15) 3.12.12
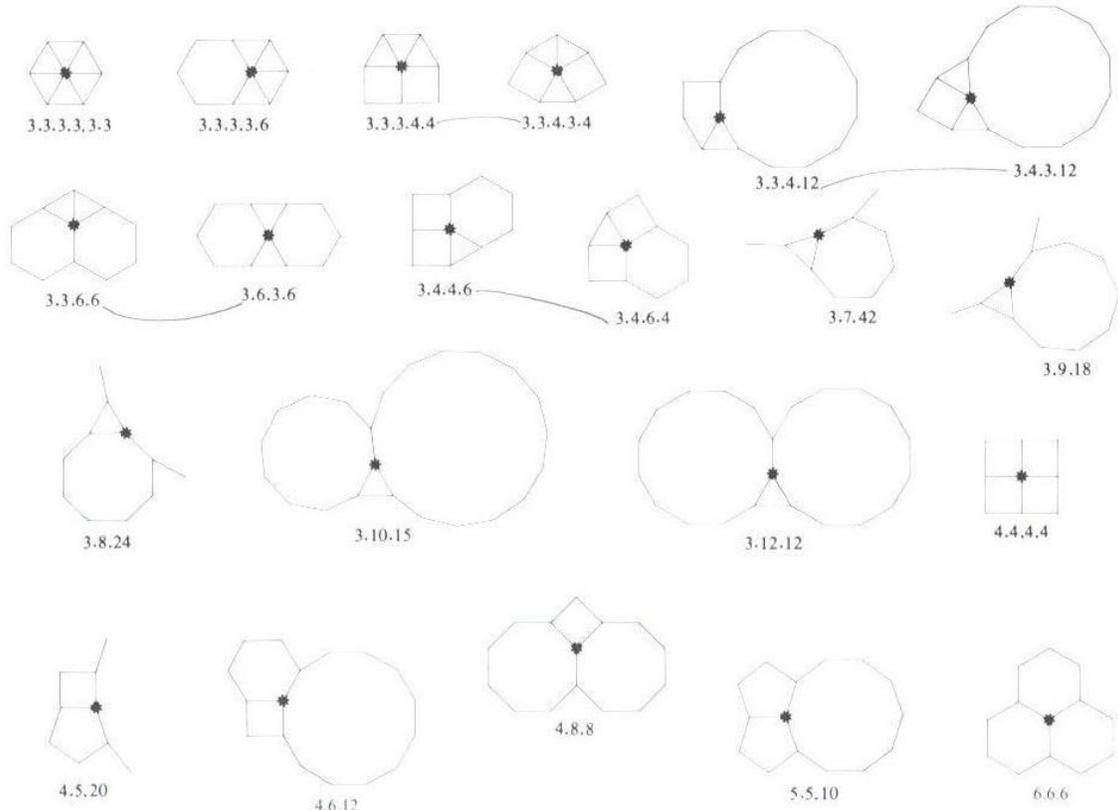16) 4.5.20

17) 4.6.12
18) 4.8.8
19) 5.5.10
20) 6.6.6

Figure 2.1.1
The 21 types of vertices possible with regular polygonal tiles.
These correspond to the last column of Table 2.1.1.

TILINGS BY REGULAR POLYGONS AND STAR POLYGONS

The picture ([1]) shows the 21 possible vertex tilings. However, it can be shown that of these vertex-tilings, only the 11 Platonic and Archimedean tilings can be extended to tile the plane. I will just give one example ([2]). Consider species of type 3.x.y.
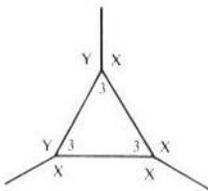
The top vertex is of type 3.x.y. In order for the left vertex to be of the same type, the bottom polygon must be an x-gon. But then the right vertex is of type 3.x.x, so we must have x=y. So from the list, species 8, 9, 10 and 11 are not possible. We can eliminate the other cases in a similar way.

We are only considering tilings that are *vertex transitive* in that given two vertices, there is a symmetry of the tiling that takes one to the other. This means that all the vertices have the same type and look the same. In a similar way, we can define a tiling to be *edge transitive* or *tile transitive* if all the edges or all the tiles look the same. The Platonic tilings are tile transitive, but not the Archimedean tilings, since they have two types of tiles. The Platonic tilings are edge transitive, but (3.6.3.6) is the only edge-transitive Archimedean tiling. In (3.6.3.6), every edge links a triangle and a hexagon, while in for instance (4.8.8), some edges link squares and octagons, while other edges link two octagons.

References

[1]   Dominique Clévenot: Splendors of Islam, Vendome Press, 2000.
[2]   Grünbaum and Shephard: Tilings and Patterns.
[3]   Kappraff: Connections
[4]   Eva Wilson: Islamic Designs for Artists and Craftspeople.

http://www.uwgb.edu/dutchs/symmetry/symmetry.htm