Undergraduate Research Opportunity Programme in Science

Polyhedra

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1. Introduction

A polyhedron is a three-dimensional figure made up of polygons. Since polygons form the basis of a polyhedron, we first discuss polygons. We need to introduce some basic terms related to polygons and polyhedra.

1.1 Basic terms

When discussing polygons we will use the terms sides and corners as shown in Figure 1.1. When discussing polyhedra we will use the terms faces, edges and vertices. These are listed below and illustrated in Figure 1.2.

1. Each polygonal part of the polyhedron is called a face.
2. A line segment along which two faces come together is called an edge.
3. A point where several edges and faces come together is called a vertex.

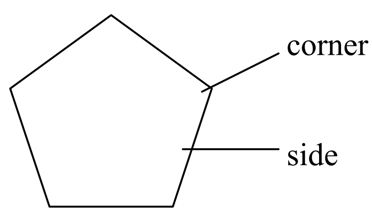


Figure 1.1. Polygon

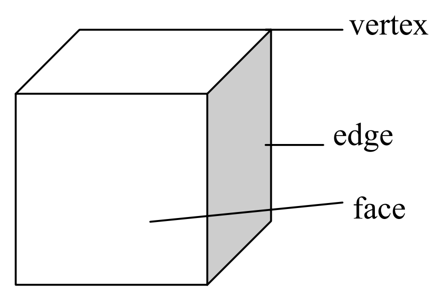


Figure 1.2. Polyhedron

1.2 What is a polygon?

A polygon is a plane figure that consists of three or more straight line segments. The line segments that form a polygonal figure must be connected. And the polygon must be a closed figure, meaning the ends of each line segment must be joined by the ends of another line segment.

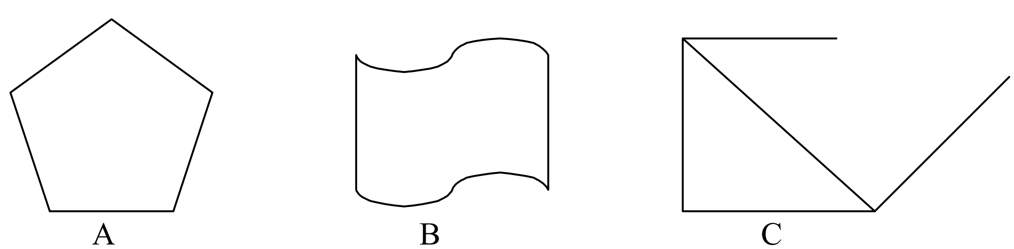
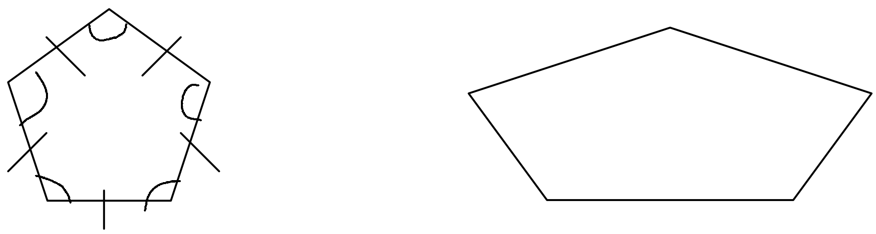


Figure 1.3. A is a polygon. B and C are not polygons.

There are two kinds of polygons, regular polygons and irregular polygons. Very simply, a polygon with all equal sides and equal angles is called a regular polygon. Otherwise it is called an irregular polygon.



Regular pentagon | Irregular pentagon

Figure 1.4. Regular and irregular polygons.

When naming a polygon, we use the Greek name for the number of sides of the polygon. For example, a five-sided polygon is called a pentagon because penta in Greek means five. Figure 1.4. shows a regular pentagon and an irregular pentagon.

Greek	Number of sides	Regular polygon	Irregular polygon
Tri	3	Equilateral triangle	Triangle
Tetra	4	Square	Quadrilateral
Penta	5	Pentagon	Irregular pentagon
Hexa	6	Hexagon	Irregular hexagon
Hepta	7	Heptagon	Irregular heptagon
Octa	8	Octagon	Irregular octagon
Nona	9	Nonagon	Irregular nonagon
Deca	10	Decagon	Irregular decagon

Table 1.1. Names of some polygons.

Earlier we learned that a polygon can be divided into two kinds, regular polygons and irregular polygons. A regular polygon itself can be divided into the following two types.

1. The fundamental or primary polygons
The sides of the primary polygons do not cross. Thus, they are convex polygons. Some examples are the regular pentagon and the regular hexagon.

2. The stellated polygons
The stellated polygons can be derived from the primary polygons by extending non-adjacent sides until they intersect. Only those that can be traced in a single line are included under stellated polygons while compound polygons are not included.

In Figure 1.5 the black lines are the sides of the primary polygons. The red lines are the extended sides of the primary polygons. In D the extended sides of a pentagon form a five-sided star called a pentagram. In A the extended sides of an octagon form an eight-sided star. The red sides of the pentagram and the eight-sided star are continuos. Thus, A and D are stellated polygons or star polygons. The red lines that are the extensions of the hexagon in B are not continuos. In B the red lines form a compound polygon. C consists of two squares on top each other. Thus, C is a compound polygon.

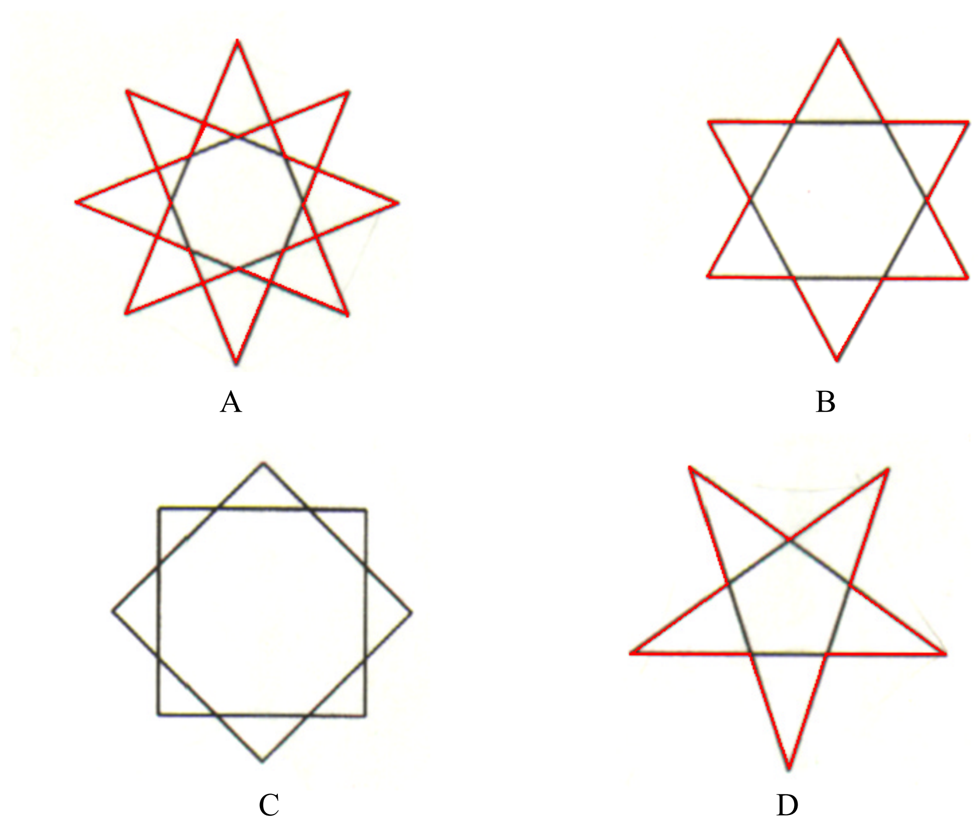


Figure 1.5. Stellated polygons A, D and compound polygons B, C.

The stellated polygons or star polygons have equal sides and angles. Thus, they are classified under regular polygons. Unlike the primary polygons, the stellated polygons are concave.

2. Polyhedra

2.1 Introduction to the Platonic solids

The Platonic solids are the tetrahedron, cube, octahedron, icosahedron and dodecahedron. The name 'Platonic' solid was derived from the name of the great Greek philosopher Plato, who wrote about them in about 400B.C. though they were known before Plato. The ancient Egyptians knew four of the Platonic solids: the tetrahedron, octahedron and cube are found in Egyptian architectural designs, and Egyptian icosahedral dice are to be found in an exhibit in the British Museum. All five were studied by the early Pythagoreans before the time of Plato and Euclid. It is in the *Elements* of Euclid, however, that we find the most extensive treatment of the geometry of these five solids.

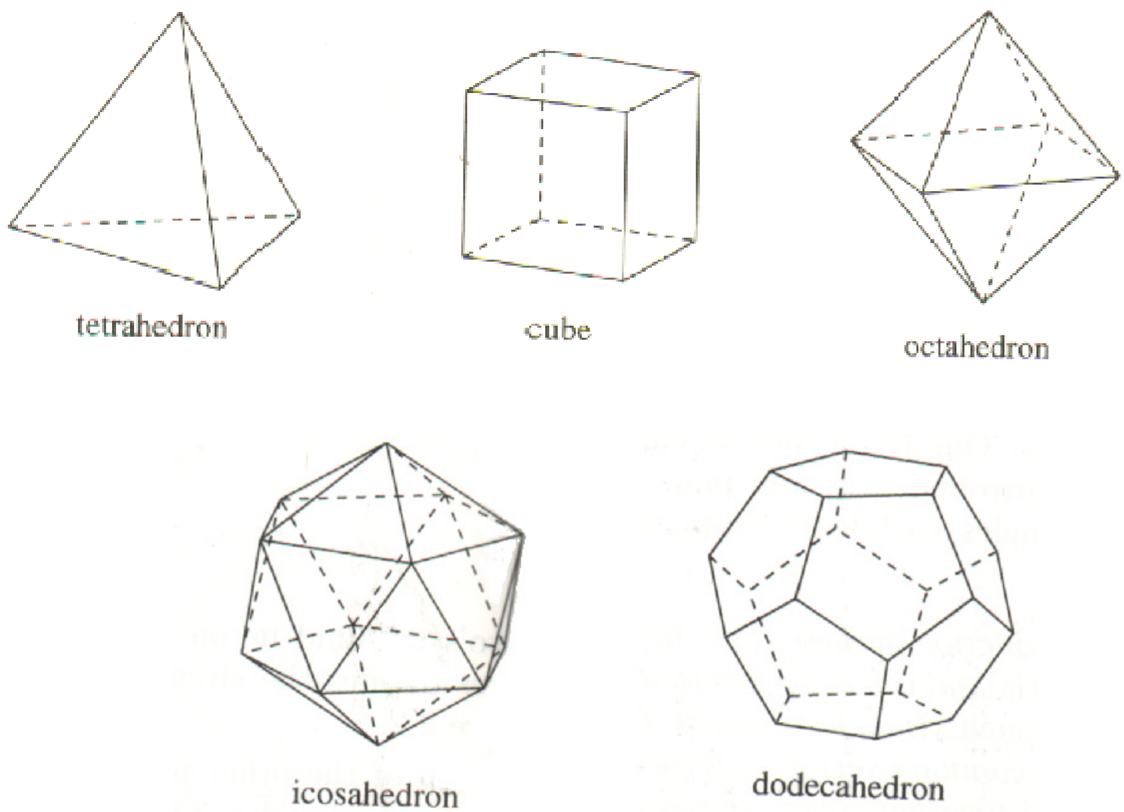


Figure 2.1. Five Platonic solids.

2.2 Introduction to the deltahedra

The convex deltahedra are the polyhedra that are made of equilateral triangles. They are called *deltahedra* because an equilateral triangle looks like the Greek capital letter *delta*, Δ.

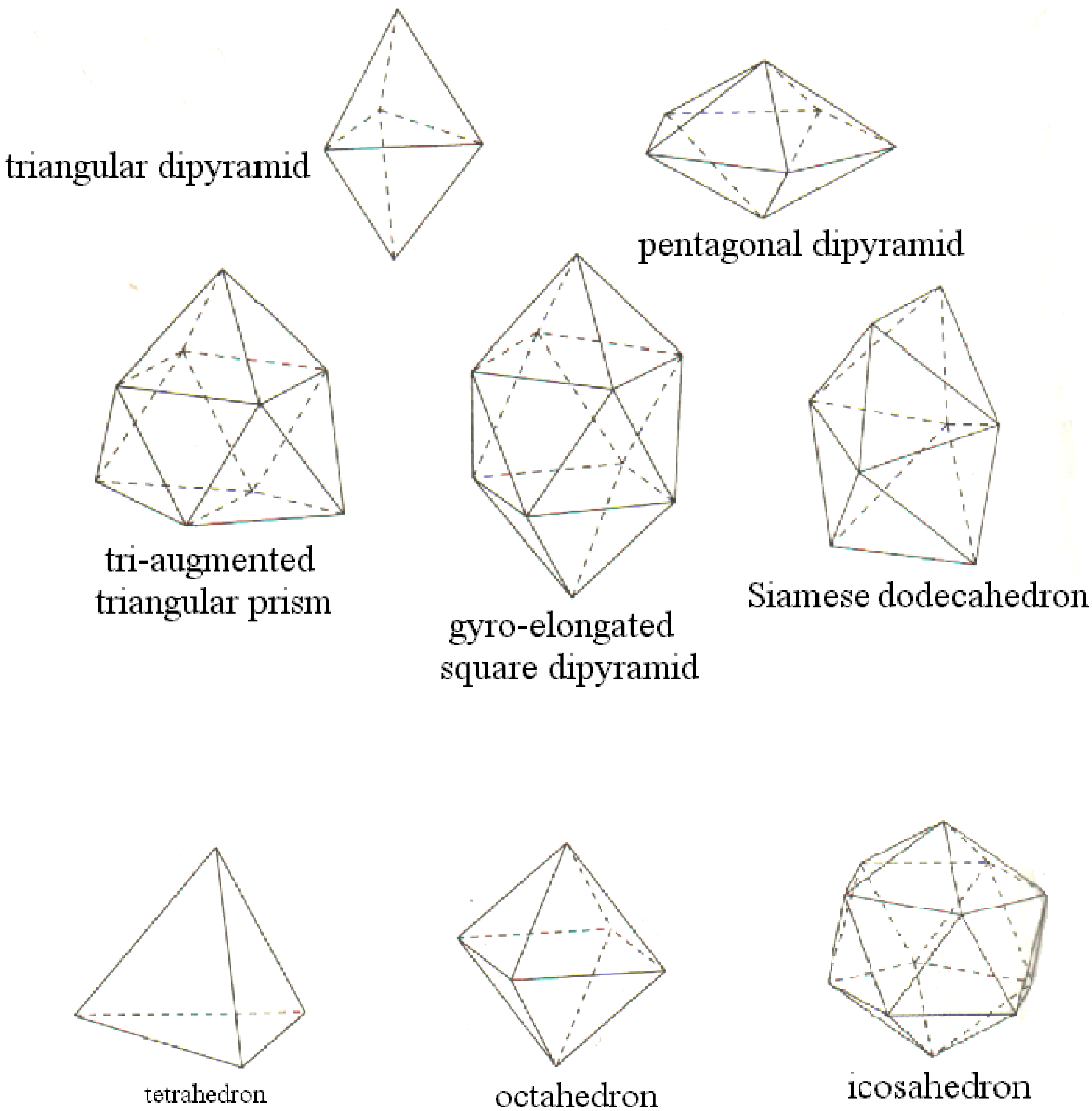


Figure 2.2. Deltahedra

2.3 Defining regular polyhedra

We define a regular polyhedron using the following two conditions.

(i) The faces must be congruent.

(ii) The faces must be regular polygons.

Both the Platonic solids and the family called the deltahedra satisfy the two conditions.

We need to modify the definition of regular polyhedron to exclude the deltahedra. The modified definition of the regular polyhedron was given by Cromwell. He stated in his book "Polyhedra" the definition given below.

A regular polyhedron, P, is a polyhedron whose faces are congruent regular polygons, and that satisfies the following equivalent conditions.

1. The vertices of P all lie on a sphere.

2. All the dihedral angles of P are equal.

3. All the vertex figures are regular polygons.

4. All the solid angels are congruent.

5. The same number of faces surrounds all the vertices.

The proof showing the equivalence of the five conditions can be obtained from the book "Polyhedra" where the definition is also found. Here are the brief descriptions of some of the basic terms that were used in Cromwell's definition. Figure 2.3 illustrates the basic terms.

(ii) Dihedral angle is the angle between two adjacent planes or the angle made by two adjacent faces. For example, the dihedral angle of a cube, which is the angle made by two adjacent square faces in a cube, is 90 degrees.

(ii) Vertex figure is the face that arises after slicing off a corner in a way that removes the same amount of each edge. For example, the vertex figure of a cube is an equilateral triangle.

(iii) Solid angle is the region of the polyhedron near a vertex. It is a chunk of the corner and is bounded by three or more plane angles. The angle in the corner of a polygonal face is called a plane angle.

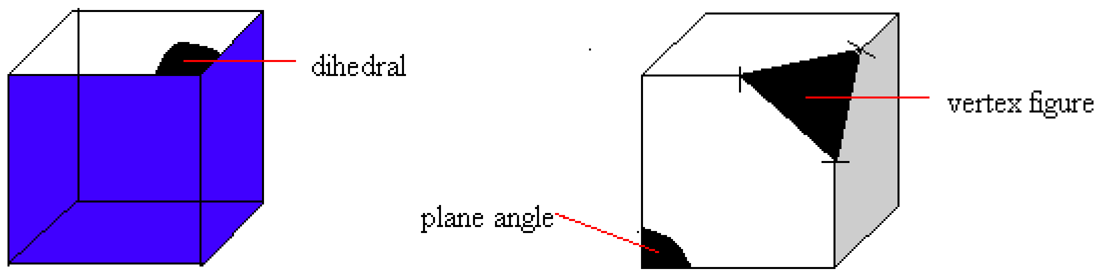


Figure 2.3. Illustrations of basic terms.

Clearly, deltahedra do not satisfy condition five in Cromwell's definition of regular polyhedra. Since the conditions are equivalent, if one condition is not satisfied all the other conditions are also not satisfied. Thus, Cromwell's definition excludes deltahedra and only includes the Platonic solids.

Following is the proof that there are only five Platonic solids.
Let us first introduce the notation {n,m}, which designates a polyhedron whose faces have n sides with m faces meeting at each vertex. For example, the notation for the cube is {4,3}. As can be seen from Table 2.1., entries in the first column are at least three. This is because every polygon has at least three sides. Also, entries in the second column are at least three, which is due to the fact that every vertex has at least three faces meeting as two would not enclose any volume. {3,6} is a flat figure because it consist of equilateral triangles with six of them at each vertex and thus, every vertex is 60 degrees X 6 = 360 degrees. {3,7}, {3,8}… have more than 360 degrees at each vertex, which does not give a convex solid. {4,4}, {4,5}, {4,6}…have more than 360 degrees at each vertex. {5,4}, {5,5}, {5,6}…also have more than 360 degrees at each vertex. {6,3}, {6,4}…also have more than 360 degrees at each vertex. In this way, all possibilities are counted and only five possibilities are left. These five possibilities are the Platonic solids.

Polyhedron	Edges on Each face	Faces at each vertex	F	V	E
Tetrahedron	3	3	4	4	6
Cube	4	3	6	8	12
Octahedron	3	4	8	6	12
Icosahedron	3	5	20	12	30
Dodecahedron	5	3	12	20	30

(F – face, V- vertex, E – edge)

Table 2.1. Brief details of the five Platonic solids.

There are only eight deltahedra as seen in Figure 2.2 but there “could be” nine. Holden in his book “Shapes, Spaces and Symmetry”, gave a simple proof to show that there could be nine deltahedra. Following is the proof that there could be nine deltahedra given by Holden. Each triangle has three sides, and therefore the triangles that make a deltahedra with N faces have 3N sides altogether. When the triangles join to make a deltahedron, each side joins a side of another triangle to make an edge, and therefore the deltahedron has 3N/2 edges. Since “half an edge” has no meaning, N must be a multiple of two. Since at least three triangles must meet at any corner, to give solidity, and at most five triangles can meet at any corner, to retain convexity, the tetrahedron is the smallest convex deltahedron and the icosahedron is the largest. And there could be seven more, with six, eight, ten, twelve, fourteen, sixteen and eighteen faces. But in fact the convex deltahedron with eighteen faces cannot be made!

Compare the two proofs given above. For deltahedra the proof gives us nine possibilities of deltahedra. However, only eight deltahedra exist because the deltahedron with eighteen faces cannot be constructed. But for Platonic solids the proof gives us five possibilities of Platonic solids and all five possibilities exist.

Similar to polygons, a polyhedron, too, can be either convex or concave. We define a polyhedron to be convex if, given any two points P and Q of the region bounded by the polyhedron, then every point of the straight line segment PQ belongs to that region. On the other hand, a concave polyhedron is simply a polyhedron that is not convex. The five Platonic solids are the convex regular polyhedra. There are also concave regular polyhedra that satisfy the modified definition of the regular polyhedra. The concave regular polyhedra are the Kepler-Poinsot solids. There are only four Kepler-Poinsot solids. Figure 2.4 shows one of the Kepler-Poinsot solids. The Kepler-Poinsot solids will be discussed in detail in Chapter 5. Thus, the regular polyhedra consist of the convex regular polyhedra and the concave regular polyhedra. Therefore, altogether there are exactly nine regular polyhedra.

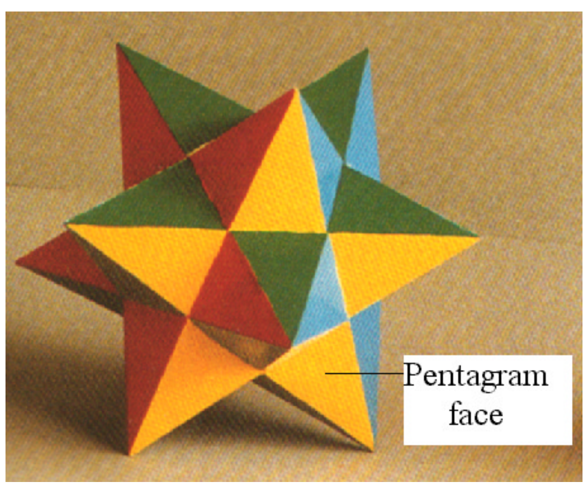


Figure 2.4. Kepler-Poinsot polyhedron with pentagram faces intersecting one another.

The Platonic solids, the deltahedra and the Kepler-Poinsot solids are not the only polyhedra present. There are many other polyhedra. Thus, this brings us to the next question- what is a polyhedron?

2.4 What is a polyhedron?

August Ferdinand Mobius gave the following definition of a polyhedron in an 1865 paper. A polyhedron is a system of polygons arranged in such a way that

(i) The sides of exactly two polygons meet at every edge.

(ii) It is possible to travel from the interior of one polygon to the interior of any other without passing through a vertex.

The first condition defines a polyhedron as a surface composed of polygons. The second condition ensures that the surface is connected. Mobius' definition of polyhedra excludes polyhedra with singularities shown in Figure 2.5. Singularities are places where the surfaces pinch together and they exist as singular vertices or singular edges. Condition (i) excludes polyhedra with singular edges and condition (ii) excludes polyhedra with singular vertices.

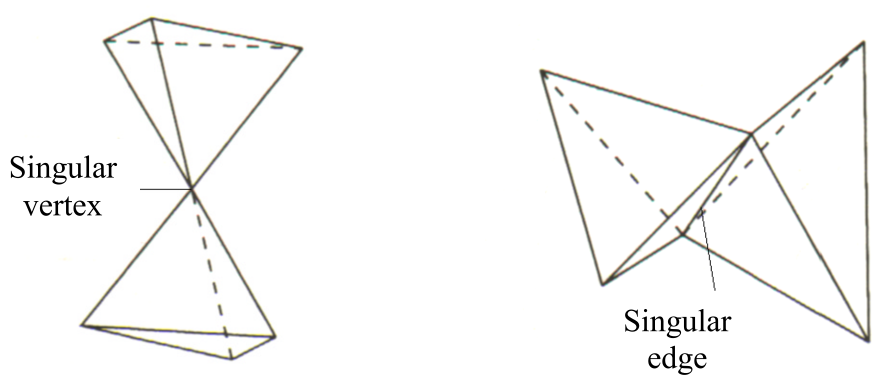


Figure 2.5. Polyhedra with singularities.

However, condition (ii) allows the polyhedral croissant shown in Figure 2.6 to satisfy Mobius' definition. Clearly, condition (ii) refers to the complete polyhedron. To exclude the 'croissant' we need to modify the definition so that it applies locally to each vertex, not globally to the polyhedron as a whole.

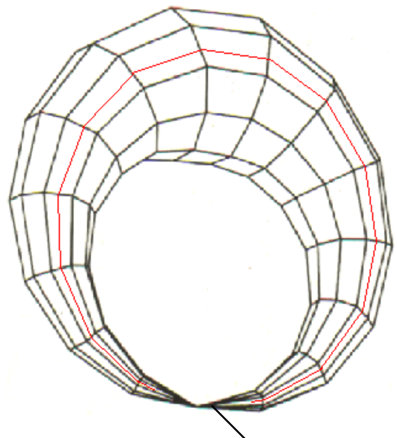


Figure 2.6. A polyhedral croissant with a red path connecting two faces at opposite ends of the singular vertex.

In his book "Polyhedra", Cromwell gave the following modified definition of a polyhedron that applies locally to each vertex.

A polyhedron is the union of a finite set of polygons such that

(i) Any pair of polygons meet only at their sides or corners.

(ii) Each side of each polygon meets exactly one other polygon along an edge.

(iii) It is possible to travel from the interior of any polygon to the interior of any other.

(iv) Let V be any vertex and let $F_1, F_2, \ldots, F_n$ be the n polygons which meet at V. It is possible to travel over the polygons F_i from one to any other without passing through V.

The first condition excludes self-intersecting polyhedra. A self-intersecting polyhedron has faces intersecting one another. Kepler-Poinsot solids come under the family of self-intersecting polyhedra. Figure 2.4. shows the self-intersecting pentagram faces of one of the Kepler-Poinsot solids. Conditions (ii) and (iv) exclude singular edges and vertices, respectively. Condition (iv) also excludes the croissant. Condition (iii) ensures that the polyhedron is connected.

In the report we will define polyhedra with Cromwell's definition but ignore the first condition in the definition throughout the report. This is to allow the self-intersecting polyhedra, to enter the family of polyhedra since Kepler-Poinsot polyhedra will be discussed in Chapter 5.

The Greeks considered the polyhedron as a solid. Later, most mathematicians considered the polyhedron as a surface or rather most mathematicians realized that it was the surface of the solid that was important. This fact is further emphasized in a statement by Euler,

"The consideration of solid bodies therefore must be directed to their boundary; for when the boundary which encloses a solid body on all sides is known, that solid is known."

3. Platonic solids

3.1 Naming of Platonic solids

Observe that the names of the five Platonic solids have the word 'hedron' at the end. Note that another name for cube is hexahedron. The term 'hedron' is a Greek word meaning "base" or "seat". The part preceding 'hedron' denotes the numbers of faces that each solid has. In Greek, tetra, hexa, octa, icosa, dodeca actually refer to four, six, eight, twenty and twelve respectively. Thus, in the case of the Platonic solids, the name tells us the number of faces, where each of it can be used as a base when a model is set on a table. For example, the tetrahedron has four faces and each of the faces can be used as a base to rest a model of it on a table.

3.2 Duals of Platonic solids

To obtain a dual we replace each edge of a polyhedron by a new edge. The new edge is perpendicular to its parent and cuts through the parent at its midpoint. In other words, the edges of a polyhedron and its dual form pairs of perpendicular bisectors. In Figure 3.1 we can see the blue edges of the cube perpendicularly bisecting the red edges of the octahedron. Thus, the cube is the dual of the octahedron and the octahedron is the dual of the cube. Generally, if A is a dual of B then B is a dual of A. We say that A and B are a dual pair.

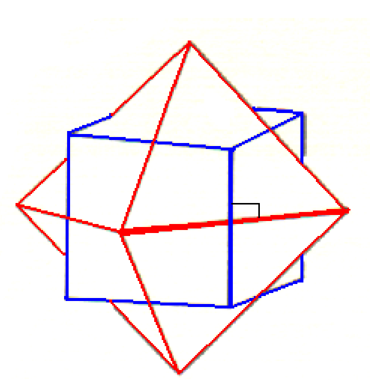


Figure 3.1. Interpenetrating dual pair-the cube and the octahedron.

For Platonic solids the cube and the octahedron are a dual pair. Also, the dodecahedron and the icosahedron are a dual pair. The tetrahedron is a dual of itself. Since the dual is formed by replacing each edge by a new edge, the number of edges remain the same for a dual pair. In Table 2.1 note that the cube and the octahedron have 12 edges each. Also, the other dual pair, the dodecahedron and the icosahedron have 30 edges each.

The perpendicular bisecting of edges causes each vertex of a polyhedron to be directly above the centre of each face of its dual and vice versa. In Figure 3.1 observe that each vertex of the octahedron is directly above each blue square face of the cube and vice versa. Thus, for a dual pair the number of faces and vertices are interchanged. In Table 2.1 notice that the cube has six faces and eight vertices while the octahedron has eight faces and six vertices. The dodecahedron has 12 faces and 20 vertices while the icosahedron has 20 faces and 12 vertices.

In Figure 3.2 the white and black Platonic solids in each row are a dual pair. The solid at the centre in each row is the result of interpenetrating the dual pair in that row. In the first row, the centre solid is the result of interpenetrating the dual pair-dodecahedron and icosahedron. Notice the vertices of the black icosahedron appearing above the white pentagonal face of the dodecahedron. The last row shows that dualizing the tetrahedron forms the tetrahedron itself.

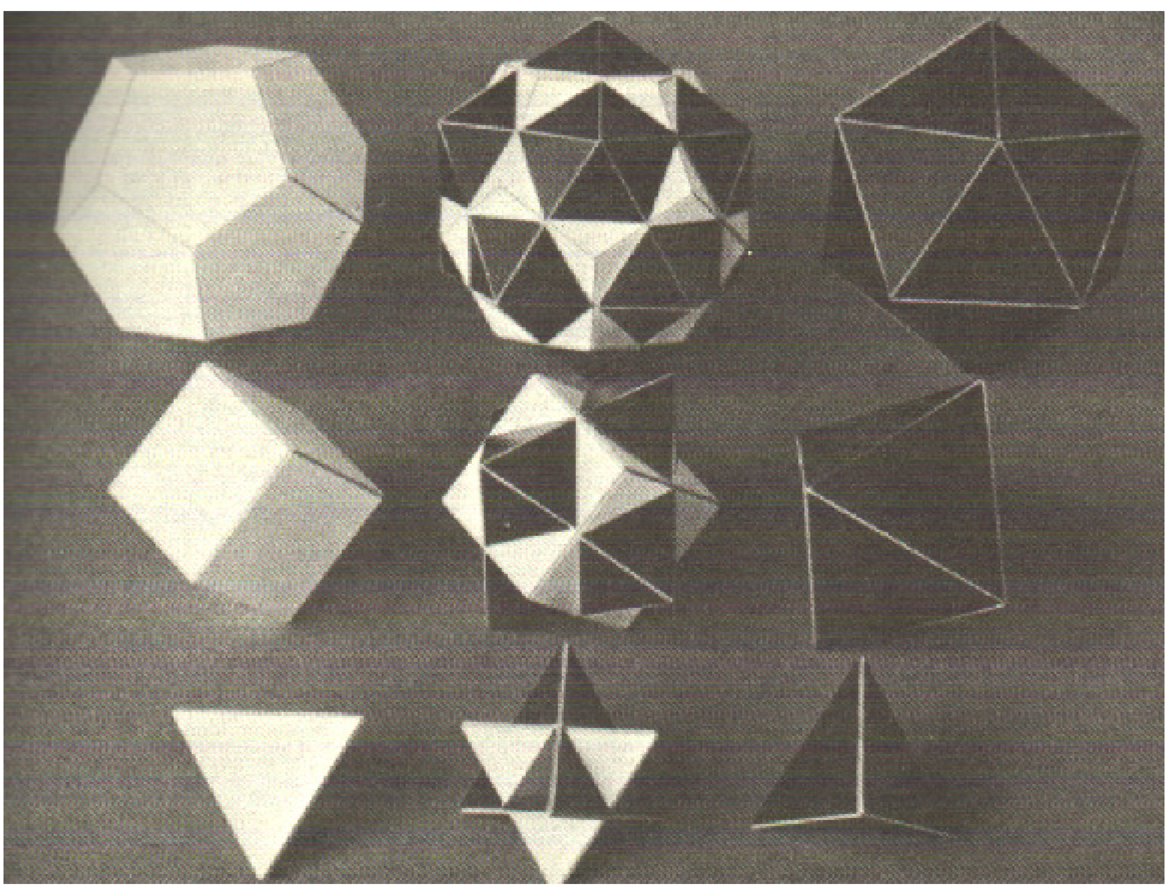


Figure 3.2. The dual pairs and their interpenetrating solids.

The dual of a polyhedron can also be formed by replacing each face of a polyhedron by a vertex. This method is similar to the edge replacing method that was mentioned earlier. Both methods when performed on a polyhedron separately, give the same dual whose number of edges remain the same but the number of faces and vertices are interchanged. However, this method of replacing each face shows clearly the change in the size of the dual that is formed after dualizing a polyhedron. Figure 3.3 illustrates this method.

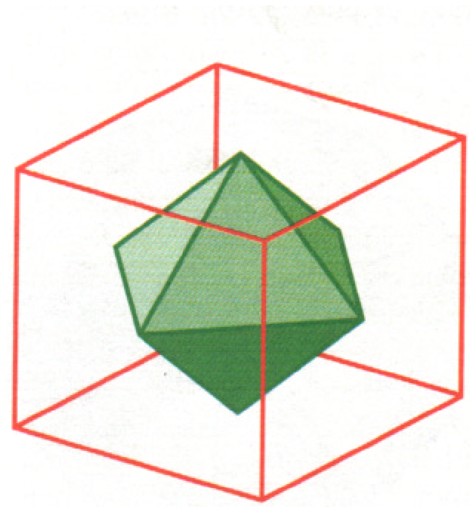


Figure 3.3. The cube and the octahedron are a dual pair.

3.3 Colouring Platonic solids

In this report we will focus on proper colouring of the Platonic solids. Proper colouring ensures that the faces that share a common edge have different colours. We will discuss the number of possible ways of proper colouring each Platonic solid by inspection since Platonic solids have a simple structure. The inspection is mainly based on the valency of the solid. Valency is the number of faces or edges that meet at a vertex. We will not consider proper colouring the dodecahedron because it is more complicated.

Tetrahedron

The tetrahedron has four faces and valency three. This forces each face to be adjacent to all the other three faces. Thus, forcing every face to have a different colour. This means that proper colouring of the tetrahedron can only be done using four colours.

With four colours we can proper colour the tetrahedron in two different ways. Place the tetrahedron on a table on one of its faces. Proper colour the tetrahedron using four different colours. Now proper colour the tetrahedron such that the base has the same colour. If the colours of the other three faces are green and blue, yellow in the clockwise direction of the vertex, then colour the three faces yellow, blue and green in the same clockwise direction. These two proper colourings are mirror images of each other thus they are said to be enantiomorphic proper colourings.

Cube

The cube has valency three. Thus, the three faces at a vertex can be properly coloured with three different colours for each face. Since the cube has six faces it can be observed that there exist only one other vertex where the remaining three uncoloured faces meet.

Colour the faces at one vertex red, blue and green clockwise around the vertex. For the other vertex surrounded by the rest of the three uncoloured faces, colour the faces red, green and blue clockwise around that vertex. This gives a proper coloured cube. Since there are only two sets of three faces each having a vertex, there is only one way to properly colour the cube using three colours

Octahedron

The octahedron has valency four. Thus, the faces at the vertex are coloured with two different colours such that adjacent faces at the vertex have different colours. The octahedron thus, can be properly coloured with two colours.

If one set of four faces at a vertex are coloured red, blue, red and blue, clockwise, the other set of four faces is coloured blue, red, blue and red, clockwise. Since the octahedron has eight faces it has only two sets of face. Thus, we can properly colour the octahedron with two colours in only one way.

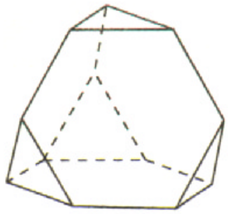
Icosahedron

The icosahedron has valency five. Thus, the faces around a vertex can only be properly coloured using three different colours and not more because each has three sides. This means that the icosahedron can be properly coloured using only three colours. Cromwell shows in his book “Polyhedra” that there are 144 combinations of the three colours that give a properly coloured icosahedron.

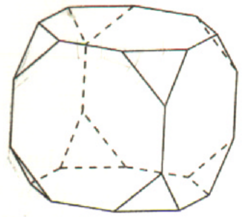
4. Archimedean solids

4.1 Introduction

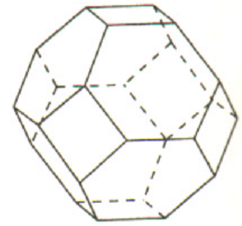
There are 13 Archimedean solids. In the fifth book of his *Mathematical Collection,* Pappus attributes the discovery of the 13 Archimedean solids to Archimedes. However, Archimedes’ own account of the Archimedean solids is lost. It was Kepler who rediscovered the whole set of 13 solids and gave them the names by which they are known today. Figure 4.1 shows the 13 Archimedean solids and their respective names.



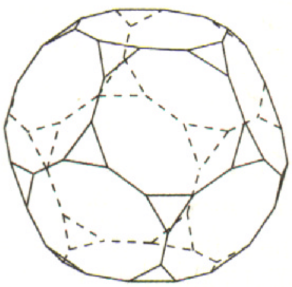
Truncated tetrahedron



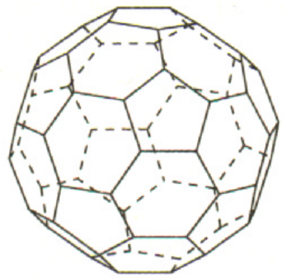
Truncated cube



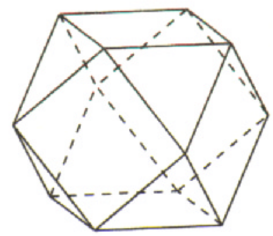
Truncated octahedron



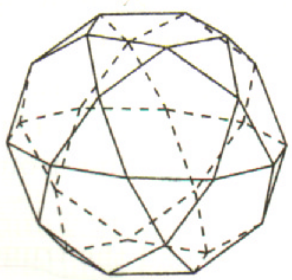
Truncated dodecahedron



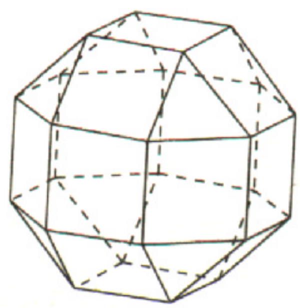
Truncated icosahedron



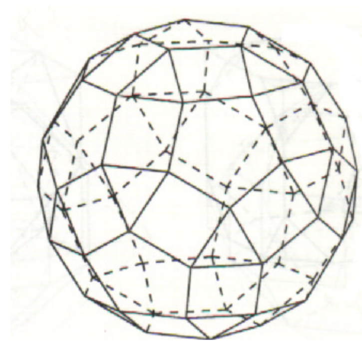
cuboctahedron



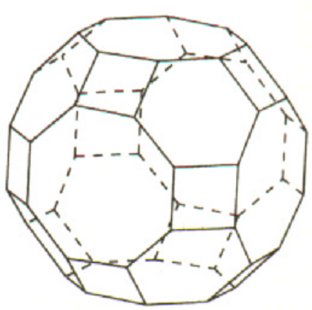
icosidodecahedron



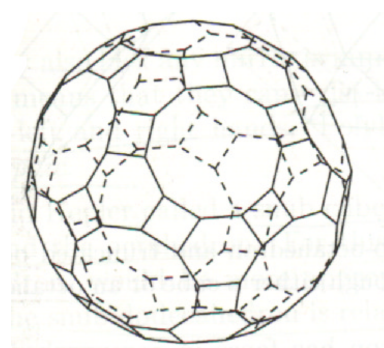
rhombicuboctahedron



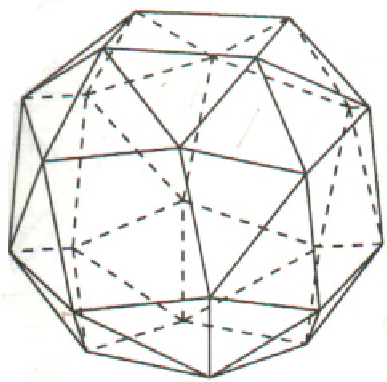
rhombicosidodecahedron



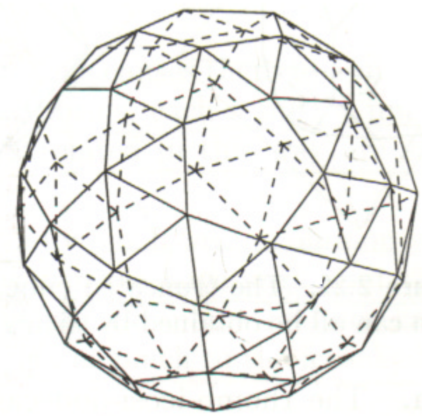
Great rhombicuboctahedron



Great rhombicosidodecahedron



Snub cube



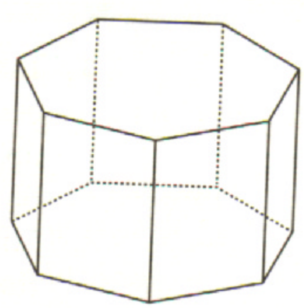
Snub dodecahedron

Figure 4.1. Archimedean solids

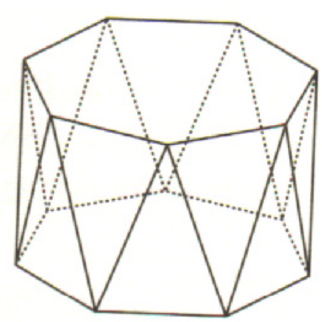
4.1.1 Semiregular polyhedra

A semiregular polyhedron is one that admits a variety of polygons as faces, provided that they are all regular and that all the vertices are congruent. A polyhedron has congruent vertices if each vertex has the same arrangement of faces. Kepler was the first to observe that two infinite sets of polyhedra, the set of prisms and the set of antiprisms, and the 13 Archimedean solids, are semiregular polyhedra.

A prism has an n-gon for top and bottom faces and rectangles around the sides. An antiprism also has an n-gon as top and bottom faces but they are rotated with respect to each other so that the vertices of the top one are between the vertices of the bottom one. The sides of an antiprism are triangles instead of rectangles. Faces of both the prism and antiprism are regular. Figure 4.2 shows an example of a prism and an antiprism.



7-gonal prism



7-gonal antiprism

Figure 4.2. Shows a prism and an antiprism.

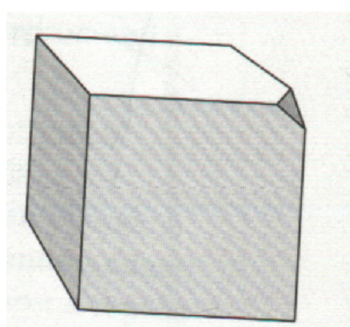
4.1.2 Construction of Archimedean solids

11 out of the 13 Archimedean solids are formed by a process called truncation. Seven of these Archimedean solids are obtained by truncating the Platonic solids. They are the truncated tetrahedron, truncated cube, truncated octahedron, truncated icosahedron, truncated dodecahedron, cuboctahedron and the icosidodecahedron. The other four Archimedean solids are the great rhombicuboctahedron, the rhombicuboctahedron, the great rhombicosidodecahedron and the rhombicosidodecahedron. These four are obtained by truncating two Archimedean solids, namely the cuboctahedron and the icosidodecahedron.

The two remaining Archimedean solids, snub cube and snub dodecahedron, are formed by a process called snubbing. As the names suggest, the snub cube is obtained by snubbing the cube and the snub dodecahedron is obtained by snubbing the dodecahedron.

4.2 What is truncation?

The process of removing all the corners of a figure in a symmetrical fashion is called truncation. What do we mean by 'symmetrical fashion'? This means that we remove a fixed fraction of the lengths of all the edges that meet at each vertex. For example, 1/3 truncation involves removing 1/3 of the lengths of all the edges that meet at each vertex. Figure 4.3 shows one corner of a cube that is truncated using a certain measurement. Truncating the remaining three corners of the cube using the same measurement gives a truncated figure.



Cube with one corner truncated

Figure 4.3. Cube with a truncated corner

Also, truncation of a figure changes the faces of the original figure and adds n sided polygons at each vertex where n edges meet. To illustrate, after truncation of the cube in Figure 4.3, the square faces of the cube change to eight sided polygons and at each vertex, where three edges meet, a triangle is obtained. Therefore,

(i) Number of faces of truncated figure = number of faces of original figure + number of vertices of the original figure

(ii) Number of vertices of truncated figure = number of vertices of vertex figure
= (number of edges at each vertex or valency of original figure) × (number of vertices of original figure)

Different measurements of truncation cause the original faces to change into different faces. Thus, truncating a particular figure using two different measurements separately results in two different truncated figures. For example, 1/3 truncation of the cube changes the original square faces to eight sided irregular polygons. ½ truncation of the cube does not change the original square faces. However, both truncations form equilateral triangles at the truncated corners. Therefore,

Let A and B be the number of faces and vertices of the original figure, respectively.

For 1/3 truncation,
Number of edges of truncated figure

= [(number of edges of changed original faces × A)
+ (number of edges of vertex figure × B)]/2
= [(number of edges of each original face × 2 × A)
+ (valency of original figure × B)]/2

For 1/2 truncation,
Number of edges of truncated figure

= [(number of edges of changed original faces × A)
+ (number of edges of vertex figure × B)]/2

= [(number of edges of each original face × A)
+ (valency of original figure × B)]/2

4.3 Archimedean solids from truncation

As we already know, Archimedean solids from truncation are obtained from truncating the Platonic solids and two of the Archimedean solids, namely the cuboctahedron and icosidodecahedron.

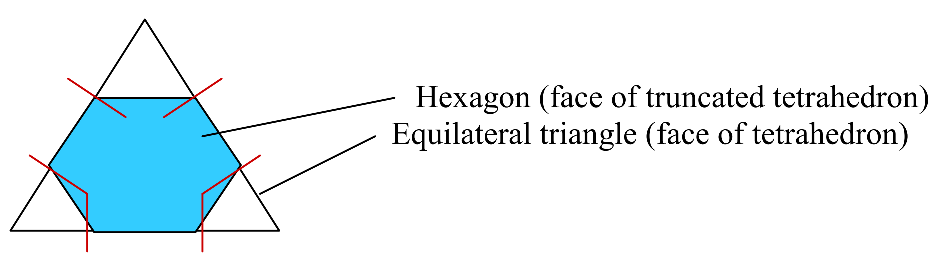
4.3.1 Truncation of Platonic solids

Table 4.1 shows Archimedean solids obtained from truncating Platonic solids.

Platonic solids	1/3 truncation	½ truncation
Tetrahedron	Truncated tetrahedron	Octahedron
Cube	Truncated cube	Cuboctahedron
Octahedron	Truncated octahedron	Cuboctahedron
Icosahedron	Truncated icosahedron	Icosidodecahedron
Dodecahedron	Truncated dodecahedron	Icosidodecahedron

Table 4.1.

1/3 truncation of the tetrahedron causes the original equilateral triangular faces to convert into regular hexagons. Since three edges meet at each vertex the vertices are converted to equilateral triangular faces that are also regular. Generally, faces formed from ½ or 1/3 truncation of Platonic solids are always regular because the original faces are regular and congruent. Figure 4.4 illustrates truncation of tetrahedron.



Red lines show 1/3 division of edges.

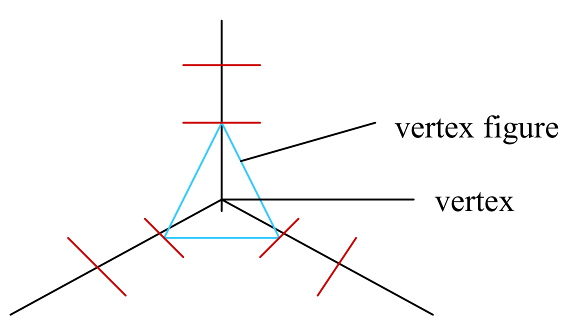


Figure 4.4. Shows faces formed on truncation of a tetrahedron

Since tetrahedron has four faces and four vertices, truncated tetrahedron consists of eight faces-four regular hexagons and four equilateral triangles. The valency of the tetrahedron is three and the number of vertices is four. Thus, the truncated tetrahedron has 4 x 3 = 12 vertices. Since this is a 1/3 truncation and the original faces are three sided, number of edges of the truncated tetrahedron is [(3 x 2 x 4) + (3 x 4)]/2 = 18. The calculations are done using the general formulas that were given earlier.

The construction of the other Archimedean solids obtained from 1/3 or ½ truncation of the Platonic solids can be explained in similar way. The number of edges, vertices and faces of all truncated Archimedean solids can be obtained in similar way. Note that using the general formulas, we will find that ½ truncation of the tetrahedron gives a Platonic solid, namely the octahedron.

Name	V	E	F3	F4	F5	F6	F8	F10
Truncated tetrahedron	12	18	4			4		
Truncated cube	24	36	8				6	
Truncated octahedron	24	36		6		8		
Cuboctahedron	12	24	8	6				
Small rhombicuboctahedron	24	48	8	18				
Great rhombicuboctahedron	48	72		12		8	6	
Snub cube	24	60	32	6				
Truncated dodecahedron	60	90	20					12
Truncated icosahedron	60	90			12	20		
Icosidodecahedron	30	60	20		12			
Small rhombicosidodecahedron	60	120	20	30	12			
Great rhombicosidodecahedron	120	180		30		20		12
Snub dodecahedron	60	150	80		12			

V- vertices, E – edges, Fn – n sided faces

Table 4.3. Numbers of vertices, edges and faces of each Archimedean solids.

Note that 1/3 truncation of the cube and dodecahedron are not exactly 1/3 truncation. This is because 1/3 truncation of square faces and pentagonal faces does not give a regular polygon. In fact it can be shown through some calculations that 1/3 truncation of the cube is actually $1 \pm \sqrt{2}/2$ truncation and 1/3 truncation of the dodecahedron is actually $[2 \pm \sqrt{(2\text{-}2\cos 108^o)}]/2(1+ \cos 108^o)$ truncation. In another words, the following measurement of truncation shown in Figure 4.5 is used to obtain regular polygons from squares and pentagons. This measurement then allows the cube and octahedron to be truncated to Archimedean solids.

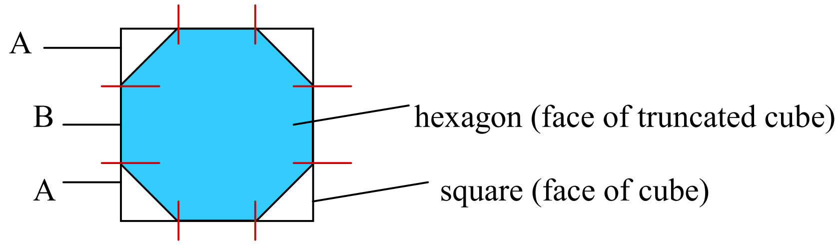


Figure 4.5. Shows measurement of truncation for cube.

In Figure 4.5 let A and B be the lengths of the portion of the edges that they represent in the figure. Therefore, truncation must be such that when each edge is divided into three parts the two parts at the ends must be equal. The two end parts are thus denoted by the same length A in Figure 4.5. The same kind of measurement is also applied to the dodecahedron to obtain the truncated dodecahedron.

4.3.2 Truncation of Archimedean solids

Table 4.2 shows the Archimedean solids obtained by truncating the cuboctahedron and the icosidodecahedron.

Archimedean solids	"1/3 truncation"	½ truncation
Cuboctahedron	Great rhombicuboctahedron	Small rhombicuboctahedron
Icosidodecahedron	Great rhombicosidodecahedron	Small rhombicosidodecahedron

Table 4.2.

1/3 truncation of the cuboctahedron and octahedron are done using the same measurements given in Figure 4.5. This is because they involve truncation of pentagonal faces to obtain regular polygons. Also, 1/3 and ½ truncation of the cuboctahedron and octahedron does not give regular polygons at the vertex, instead they give rectangles. The rectangles are adjusted to squares in order to obtain Archimedean solids. Therefore, 1/3 and ½ truncation of the cuboctahedron and the icosidodecahedron are not exactly 1/3 and ½ truncations.

4.4 Naming of Archimedean solids

Notice that 1/3 truncations of the Platonic solids and the Archimedean solids have the term 'truncated' in their name. For example, 1/3 truncation of the tetrahedron and the cuboctahedron are called truncated tetrahedron and truncated cuboctahedron, respectively. Since 1/3 truncation of the cuboctahedron is not a true 1/3 truncation, the truncated cuboctahedron is called the great rhombicuboctahedron. The prefix 'great' is used to differentiate from the rhombicuboctahedron, which is also called the small rhombicuboctahedron. Similar argument holds for the truncated rhombicosidodecahedron, which is called the great rhombicosidodecahedron.

Cuboctahedron can be obtained from ½ truncation of either the cube or the octahedron. Thus, the cuboctahedron has faces in common with the cube and the octahedron as shown in Figure 4.6. Therefore, it is called cuboctahedron. Similar argument holds for the icosidodecahedron which is formed from ½ truncation of either the icosahedron or the dodecahedron.

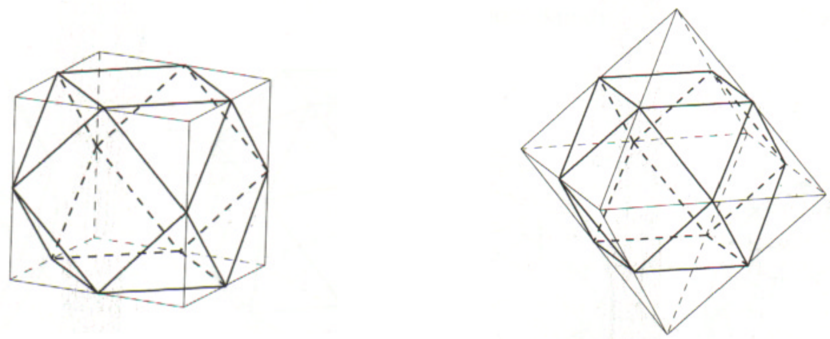


Figure 4.6.

The rhombicuboctahedron is obtained from ½ truncation of the cuboctahedron. Thus, it has faces in common with the cuboctahedron. The cuboctahedron as seen earlier has faces in common with the cube and the octahedron. Therefore, the rhombicuboctahedron has faces in common with the cube and the octahedron. The rhombicuboctahedron also has faces in common with a rhombic polyhedron, namely the rhombic dodecahedron. Therefore, since the rhombicuboctahedron has faces in common with the cube, the octahedron and the rhombic dodecahedron, it is called rhombic cuboctahedron. This is illustrated in Figure 4.7. More about the rhombic dodecahedron will be discussed in Section 4.8.1. Similar argument holds for the rhombicosidodecahedron. It has faces in common with the icosahedron, the dodecahedron and a rhombic polyhedra called the rhombic triacontahedron. More about the rhombic triacontahedron will be discussed in Section 4.8.2.

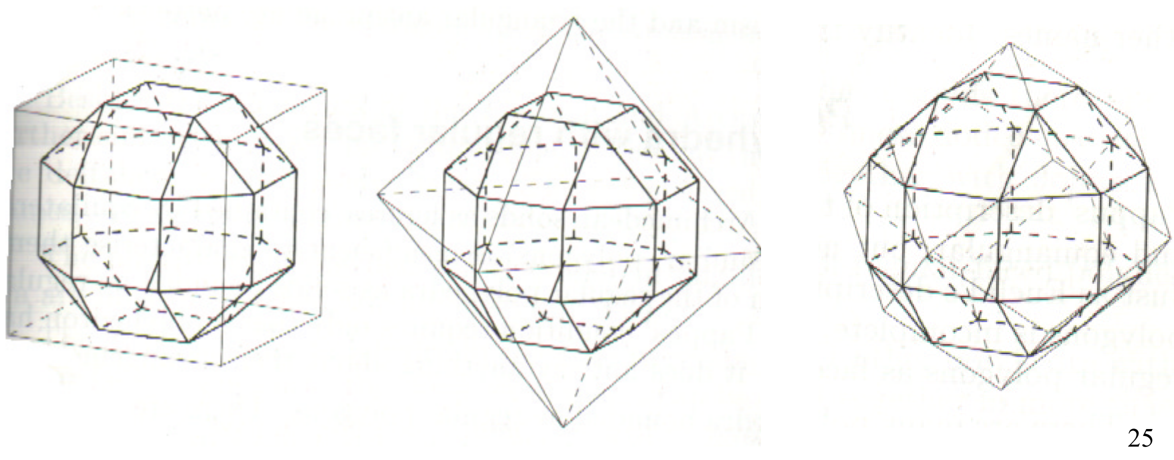


Figure 4.7

The remaining two Archimedean solids, the snub cube and the snub dodecahedron are so called because they are obtained by snubbing the cube and the octahedron, respectively.

4.5 What is snubbing?

Snubbing is a process that is used on polyhedra. Following are the three steps involved in snubbing a polyhedron.

1. Pull the faces of the polyhedron apart.

2. Replace the edges of each of the original face with pairs of triangles. The triangles can point to the left or right of each other.

3. Replace vertices where n faces meet with n-sided polygons.

The steps involved in snubbing will be explained clearly using the snub Archimedean solids as illustrations.

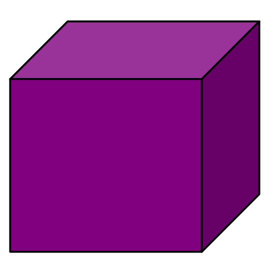
4.6 Snub Archimedean solids

The snub cube is obtained by snubbing the cube or the octahedron. The snub dodecahedron is obtained by snubbing the dodecahedron or the icosahedron. Snubbing the tetrahedron gives a Platonic solid, namely the icosahedron.

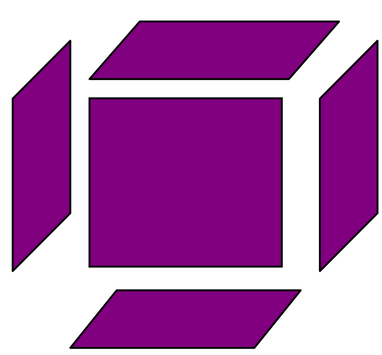
4.6.1 Snub cube

The following illustrate the three steps involved in constructing the snub cube.

1. Pull the faces of the cube apart. Thus, we have six separated squares. Figure 4.8 illustrates this step.



Cube



Faces of cube pulled apart

Figure 4.8.

2. Consider two originally adjacent faces of the cube. Attach an equilateral triangle to each face such that the triangle is attached to the edge of the face that was originally the common edge between the two adjacent square faces. Repeat the process by attaching an equilateral triangle to every edge of the square faces in the manner shown in Figure 4.9. Now move the adjacent faces towards each other as indicated by the arrows in Figure 4.9 to place the equilateral triangles side by side.

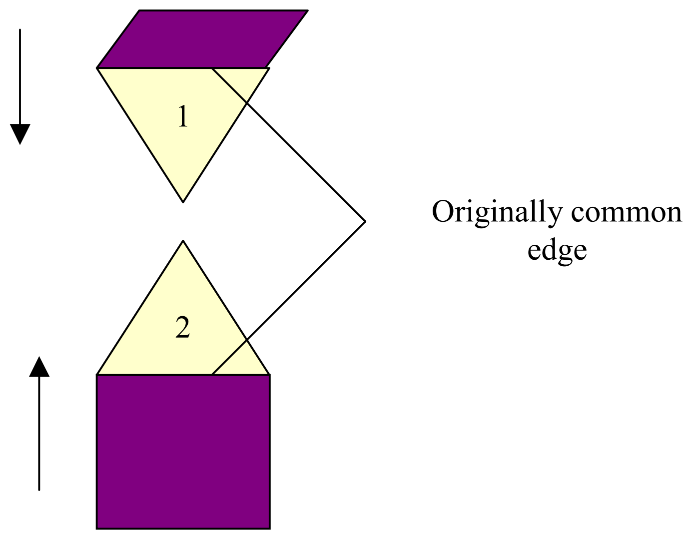


Figure 4.9. Shows how to attach the equilateral triangles.

This movement causes equilateral triangle 1 to point to the left or right of triangle 2 and in the process the square faces are tilted as shown in Figure 4.10. Thus, giving rise to two different arrangements of a pair of equilateral triangles. Throughout the whole snub cube the triangles in the pairs of equilateral triangles will either point to the left or right of each other. If the triangles point to the left it is called a left snub cube and if it points to the right it is called the right snub cube. Polyhedra related in this way are said to be enantiomorphic. Note that equilateral triangles are used because faces of Archimedean solids are regular.

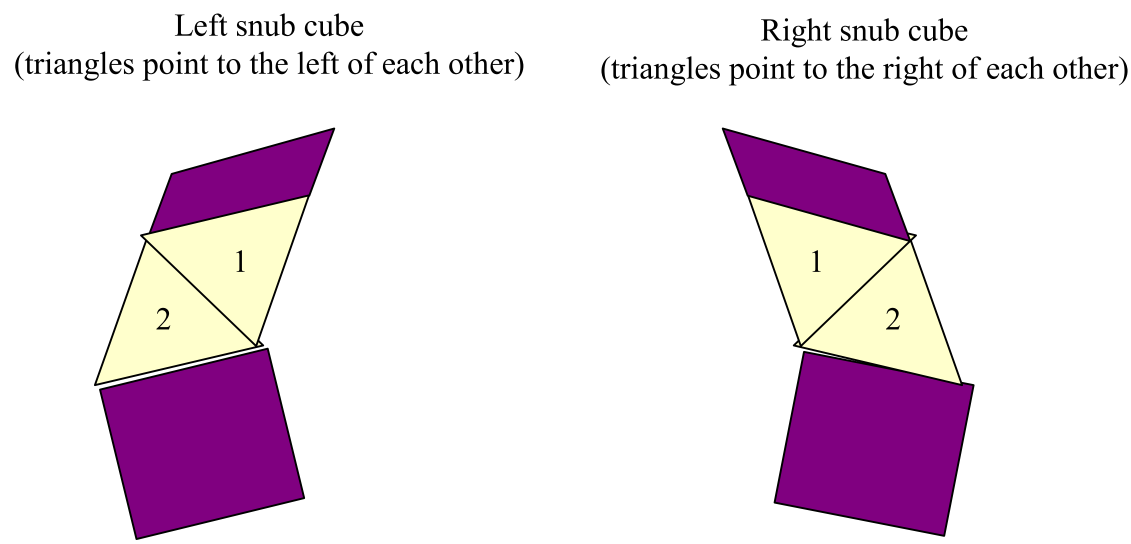


Figure 4.10. Shows the arrangement of triangles in the left and right snub cube.

How many pairs of equilateral triangles are needed to snub a Platonic solid to obtain snub Archimedean?

Let the Platonic solid consist of k n-sided faces.
Each n-sided face of the figure is adjacent to n other faces. Between each pair of adjacent faces there is a pair of equilateral triangles. Thus, each face is surrounded by n pairs of equilateral triangles. Since there are k faces there are kn pairs of equilateral triangles but each pair is shared between two adjacent faces. Therefore, Number of pairs of equilateral triangles = kn/2

Following this formula, the snub cube has 12 pairs of equilateral triangles.

3. Three square faces meet at each vertex of the cube. Pulling the faces of the cube apart forms empty spaces between the three faces sharing a common vertex. The common vertex splits into three vertices, one for each of the three faces that meet at the vertex. Place an equilateral triangle such that the vertices of the equilateral triangle meet the three vertices that were split from the common vertex. In Figure 4.11 the green triangle represents the three-gon that replaced one of the vertices where the three purple square faces meet.

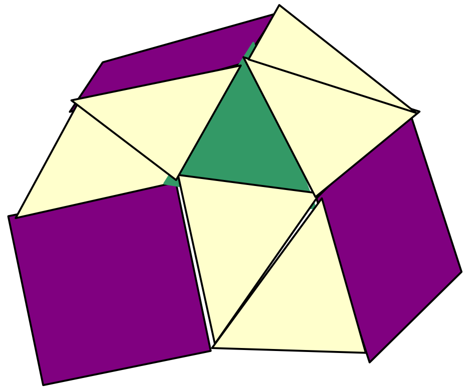


Figure 4.11. Shows triangle replacing vertex.

How many h-sided regular polygons replace the vertices of a Platonic solid to obtain a snub Archimedean?

Let the Platonic solid consist of h faces meeting at each vertex with m vertices altogether. Since the Platonic solid has h faces at each vertex, when the faces are pulled apart, each vertex splits into h vertices. The regular polygon that replaces each vertex joins h vertices to link the h faces meeting at each vertex. Thus, the regular polygon is h-sided. The h-sided regular polygon replaces all the vertices of the Platonic solid. Therefore, number of h-sided regular polygons = m

Following this formula, the snub cube has eight three-sided regular polygons which are equilateral triangles.

How many vertices does a Platonic solid that is snubbed to obtain Archimedean solid have?

Let the Platonic solid consist of k n-sided faces.
Basically, snubbing involves separating the faces of a Platonic solid and squeezing in equilateral triangles and h-sided polygons. Separating the faces forms kn vertices and squeezing of regular polygons does not create any vertices. Therefore, total number of vertices = kn

Following this formula the snub cube has 24 vertices.

How many edges does a Platonic solid that is snubbed to obtain an Archimedean solid have?

Let the Platonic solid consist of k n-sided faces. Assume the Archimedean that is obtained from snubbing the Platonic solid to have x equilateral triangles and y h-sided regular polygons. Therefore total number of edges is 3x + hy +kn. Since each edge is shared by two edges, total number of edges = (3x + hy +kn)/2

Following this formula, total number of edges of the snub cube = [3(24) + (3)(8) + (6)(4)]/ 2
= 60

The total number of edges, vertices, and faces of the snub cube and dodecahedron are shown in table 4.3. The formulas give us a faster way to find these numbers. Also, the formulas enable us to predict the type of figure formed upon snubbing.

Snub octahedron

The snub cube can also be obtained by snubbing the octahedron. The three steps are explained briefly to show that snubbing the octahedron gives the snub cube.

1. The eight equilateral triangle faces of the octahedron are separated. Thus, total number of vertices = 8 x 3 =24.

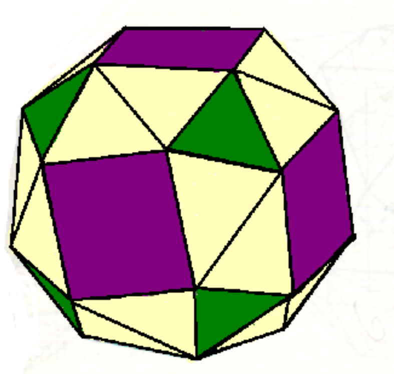
2. A pair of equilateral triangles is placed between two adjacent triangle faces. Each face of the octahedron is surrounded by three pairs and each pair is shared by two faces. Therefore, total number of pairs of equilateral triangles = (8 x 3)/2 = 12.

3. Four equilateral triangles meet at each vertex of the octahedron. These four equilateral triangles are linked by a four sided regular polygon, namely the square. Thus, each of the six vertices in the octahedron is replaced by a square. Therefore, total number of squares = 6.

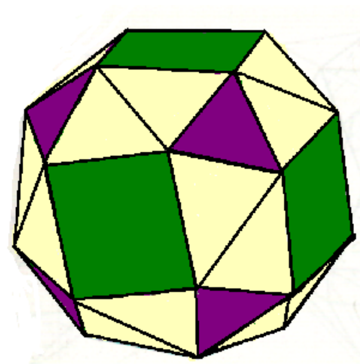
Snub cube = snub octahedron

Both the snub cube and the snub octahedron have the same number of pairs of equilateral triangle. Only the regular polygons in steps one and three are interchanged. For the snub cube six squares are separated and eight triangles replace the vertices. For the snub octahedron eight triangles are separated and six squares replace the vertices. Figure 4.12 illustrates this.

In Figure 4.12 the purple faces are faces of the Platonic solid that is snubbed. The green faces are the faces that replace the vertices in the Platonic solid. The yellow triangles are the pair of equilateral triangles.



Left snub cube



Right snub octahedron

Figure 4.12. Shows that the snub cube and snub octahedron are the same polyhedra.

The fact that the snub cube and the snub octahedron are the same polyhedra is not surprising because the cube and the octahedron are duals. The property of duals is that their faces and vertices are interchanged. The cube has six square faces separated and eight triangular vertex figures while the octahedron has eight triangular faces and six square vertex figures. Since both the snub cube and the snub octahedron consist of 24 vertices, 32 equilateral triangles and six squares, they must have the same number of edges, too. Thus, the snub cube and the snub octahedron are the same figures.

It therefore follows that the six squares on the snub cube are faces of the cube. The eight equilateral triangles that are not pairs of equilateral triangles in the snub cube are the faces of the octahedron.

Enantiomorphism of the snub cube and the snub octahedron

The left snub cube is the right snub octahedron and the right snub cube is the left snub octahedron. This is because to identify the left and right snub octahedron we consider the arrangement of the pair of equilateral triangles in relation to the face of the Platonic solid that is snubbed.

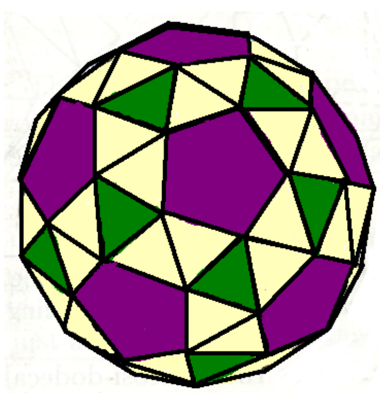
This means that in Figure 4.12 we consider the direction in which the yellow triangle (equilateral triangles) attached to the purple face (faces of Platonic solid) point to with respect to its corresponding adjacent yellow triangle. The two triangles are the equilateral pair of triangles. Following this argument we will find that the snub cube in Figure 4.12 is a left snub cube and the snub octahedron is a right snub octahedron. Since both polyhedra are the same we can see that the left snub cube is equal to the right snub octahedron.

4.6.2 Snub dodecahedron

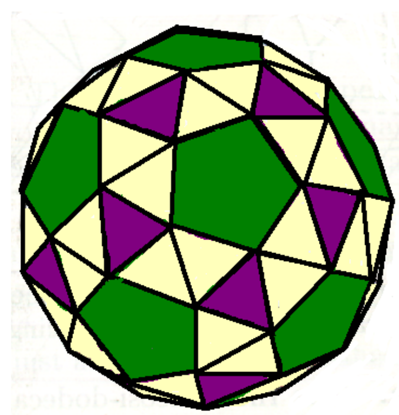
The steps involved in snubbing the dodecahedron are similar to the steps involved in snubbing the cube. Snubbing the duals, the cube and the octahedron gives the same polyhedra, namely the snub cube. Therefore, snubbing the duals, the dodecahedron and the icosahedron also gives the same polyhedra called the snub dodecahedron. This is illustrated in Figure 4.13 in the similar manner as Figure 4.12.

	Snubbing the dodecahedron	Snubbing the icosahedron
Step 1	**12** pentagonal **faces** are separated. Total number of vertices = 12 x 5 = 60	**20** equilateral triangular **faces** are separated. Total number of vertices = 20 x 3 = 60
Step 2	Each pentagonal face is surrounded by five pairs of equilateral triangles. Each pair is shared by two faces. Total number of pairs of equilateral triangle = (12 x 5)/2 = 30	Each triangular face is surrounded by three pairs of equilateral triangles. Each pair is shared by two faces. Total number of pairs of equilateral triangle = (20 x 3)/2 = 30
Step 3	Dodecahedron has **20 vertices**. Replace each of the vertices where three pentagonal faces meet with equilateral triangles. Total number of nonpaired equilateral triangles = 20	Icosahedron has **12 vertices**. Replace each of the vertices where five triangular faces meet with regular pentagons. Total number of pentagons = 12
	Snub dodecahedron has 80 F3, 12 F5 and 60V.	Snub icosahedron has 80 F3, 12 F5 and 60V.
Same composition of faces gives same number of edges. Therefore, snub dodecahedron = snub icosahedron		

Table 4.4. Shows snubbing of dodecahedron and icosahedron



Left snub dodecahedron



Right snub icosahedron

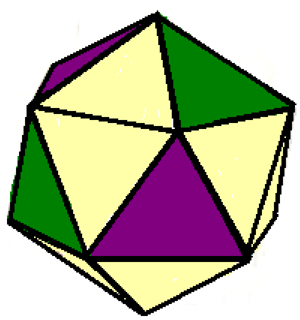
Figure 4.13. The snub dodecahedron and the snub icosahedron are the same polyhedra.

Similar rules are used to identify the left and right snub dodecahedron. Also, the left snub dodecahedron is the right snub icosahedron and the right snub dodecahedron is the left snub icosahedron. Figure 4.13 shows a left snub dodecahedron and a right snub icosahedron.

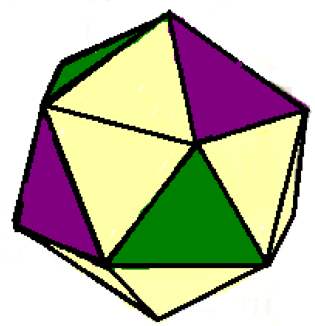
4.6.3 Snub tetrahedron

We can use the steps involved in snubbing and the formulas used to obtain the number of vertices, faces and edges to verify that the snub tetrahedron is the icosahedron.

Both the left and the right snub tetrahedron are contained in one icosahedron or snub tetrahedron. This is because in the icosahedron two different sets of four equilateral triangles can be used to represent the faces of the tetrahedron. One of the two sets corresponds to the right snub tetrahedron and the other corresponds to the left snub tetrahedron. Figure 4.14 shows the left and the right snub tetrahedron in the icosahedron.



Left snub tetrahedron



Right snub tetrahedron

Figure 4.14.

4.7 Isomers of Archimedean solids

Isomers are polyhedra with different arrangement of the same faces. The Archimedean solids with regular polygons for “equators” have isomeric forms. This is because when we cut on the equator of an Archimedean solid, twisting the two halves or caps and putting them back together gives a different arrangement of the same faces. Archimedean solids with isomers are cuboctahedron, icosidodecahedron, rhombicuboctahedron and rhombicosidodecahedron. Figure 4.15 shows the Archimedean solids with isomeric forms.

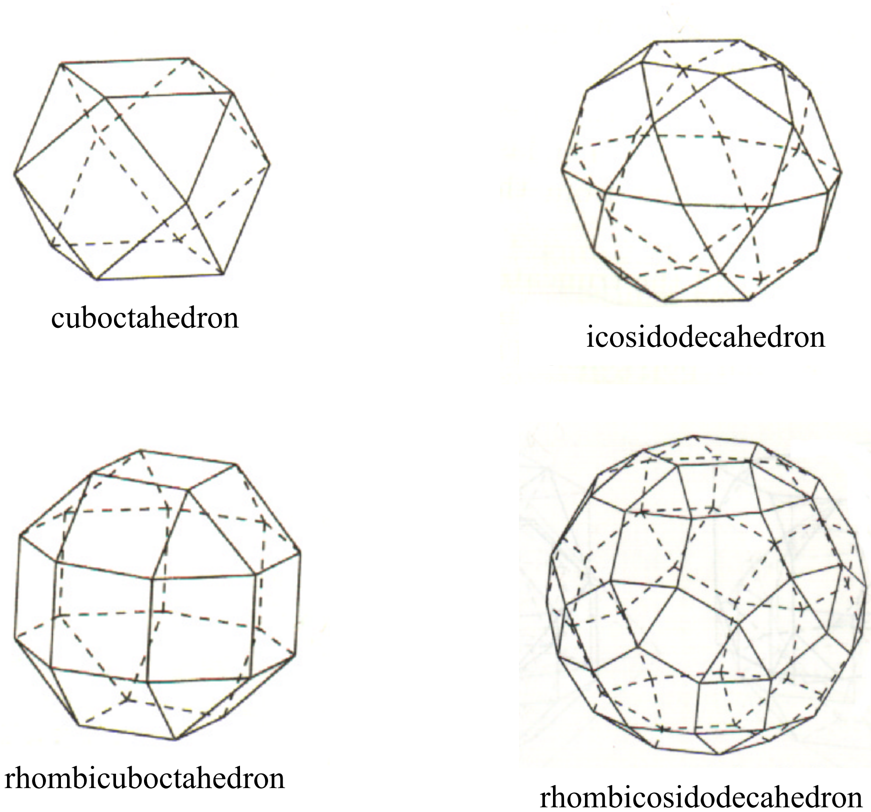


Figure 4.15. Shows equators of Archimedean solids with isomers.

The cuboctahedron has a regular hexagon for equator. An isomer is possible by twisting one cap 60 degrees. The icosidodecahedron has a regular decagon for equator. Thus, an isomer is possible by twisting one cap 36 degrees. Both the isomers of the cuboctahedron and the icosidodecahedron give a different arrangement of the same faces with some triangles adjacent to triangles. The rhombicosidodecahedron has more than one regular decagonal equators. Several isomers can be formed in this case by twisting various caps by 36 degrees:

(i) twist a single cap,
(ii) twist two opposite caps,
(iii) twist two non-opposite caps
(iv) twist three caps.

None of the isomers of the cuboctahedron, icosidodecahedron and rhombicosidodecahedron are Archimedean solids. This is because these isomers do not have congruent vertices. As we mentioned earlier in Section 4.1.1, two vertices are congruent if they have the same arrangement of faces.

The rhombicuboctahedron has a regular octagon for equator. An isomer is obtained by twisting one cap 45 degrees. This isomer is called Miller's solid that is shown in Figure 4.16. Miller's solid is the most important isomer of all the Archimedean isomers because it has congruent vertices like the Archimedean solids while the other isomers do not have congruent vertices. Because of this, some writers have suggested that this polyhedron should be counted as a fourteenth Archimedean solid. However, Miller's solid still cannot be included as an Archimedean. This is because if we define the Archimedean solids with an extra condition that says that Archimedean solids are vertex-transitive then Miller's solid is not an Archimedean solid under the modified definition.

A polyhedron is vertex-transitive if any vertex can be carried to any other by a symmetry operation. This corresponds to the fact that a polyhedron looks the same when viewed with any of it vertices directed forward. The Archimedean solids are all vertex-transitive. However, Miller's solid has four-fold rotational symmetry with four rotational symmetry axis in the belt of eight squares. Thus, causing Miller's solid to have 16 symmetries but they are insufficient to carry a vertex onto each of the 23 others. Note that Miller's solid has a vertical mirror. More details on the symmetry group of Miller's solid and Archimedean solids can be found in the book "Polyhedra" by Cromwell. Visually in Figure 4.16 we can see that the presence of three belts of squares makes the rhombicuboctahedron vertex-transitive. While the presence of only one belt of squares makes Miller's solid non vertex-transitive.

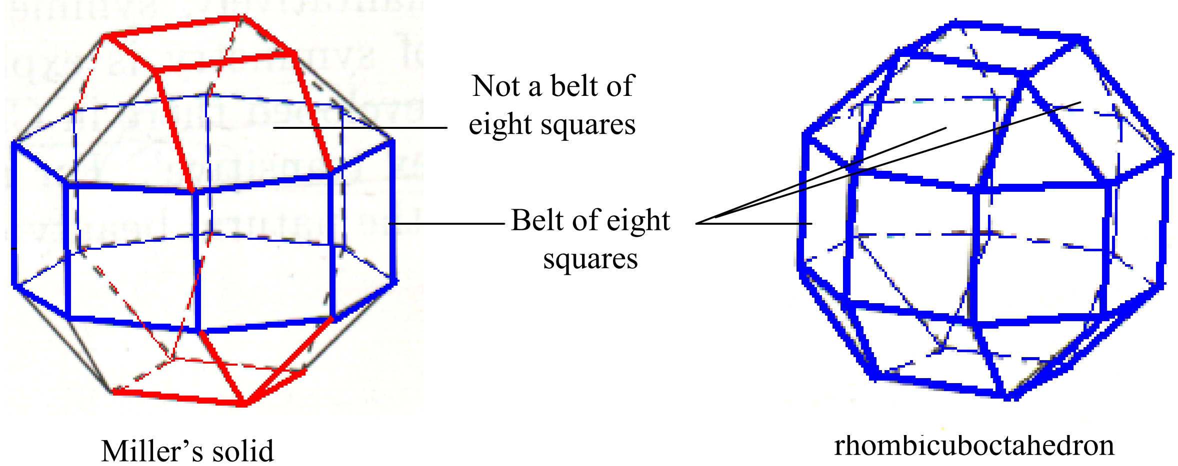


Figure 4.16. Shows some difference in the structure of the rhombicuboctahedron and Miller's solid.

4.8 Duals of Archimedean solids

Archimedean duals are obtained in the same way as Platonic duals. Each of the Archimedean solids has more than one type of regular face but congruent vertices. Therefore, their duals have more than one type of vertex but only a single type of irregular face. Similar to Platonic duals, in Archimedean duals the number of vertices and faces are interchanged. We will only discuss duals of the cuboctahedron and icosidodecahedron. These two duals are the only Archimedean duals with rhombic faces.

4.8.1 Rhombic dodecahedron

The dual of the cuboctahedron is known as rhombic dodecahedron and it is shown in Figure 4.17 since the corners of the cuboctahedron are all alike, the faces on the rhombic dodecahedron are all alike. Dualizing causes each vertex to be replaced by a face. Since the cuboctahedron, has valency four the rhombic dodecahedron has quadrilateral faces. The arrangement of faces around each vertex in the cuboctahedron is (3,4,3,4), meaning a triangle, a square, another triangle and another square meet at each vertex. Since opposite faces at each vertex is the same, the quadrilateral faces of the rhombic dodecahedron are rhombic. The cuboctahedron has 12 vertices. Therefore, its dual has 12 rhombic faces.

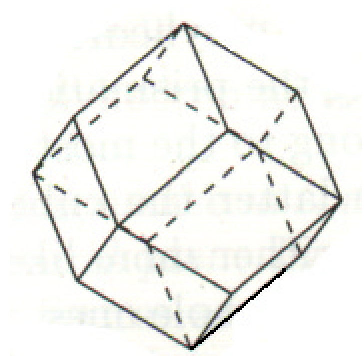


Figure 4.17. A rhombic dodecahedron.

The cuboctahedron consists of two types of faces, squares and triangles. Thus, the rhombic dodecahedron has two types of vertices. This is because dualizing causes each face to be replaced by a vertex. Thus, three rhombi surround the vertex that replaces each triangular face and four rhombi surround the vertex that replaces each square. The cuboctahedron has 14 faces of which eight are triangles and six are squares. Therefore, its dual has 14 vertices of which eight vertices are surrounded by three rhombic faces and six vertices are surrounded by four rhombic faces.

Similar to duals of Platonic solid, the interpenetrating polyhedra of the dual pair-cuboctahedron and rhombic dodecahedron in Figure 4.18 show the vertices of the black cuboctahedron directly above the white rhombic dodecahedron and vice versa.

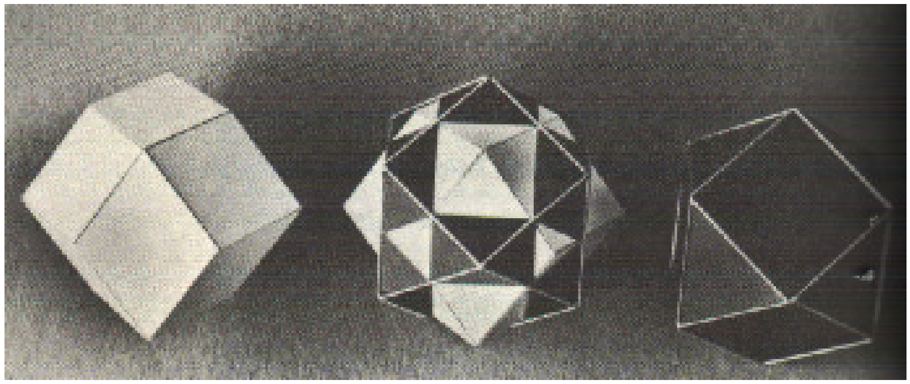


Figure 4.18. The dual pair and its interpenetrating solid.

4.8.2 Rhombic triacontahedron

The dual of the icosidodecahedron is known as the rhombic triacontahedron. Figure 4.19 shows a rhombic triacontahedron. Similar to the cuboctahedron, four faces meet at each vertex of the icosidodecahedron. Thus, the faces of the rhombic triacontahedron are quadrilateral. The arrangement of faces at each vertex is (3,5,3,5) for the icosidodecahedron. This is similar to that of the cuboctahedron because opposite faces at each vertex are the same. Thus, the faces of the rhombic triacontahedron are also rhombic. Since the icosidodecahedron has 30 vertices, its dual has 30 rhombic faces.

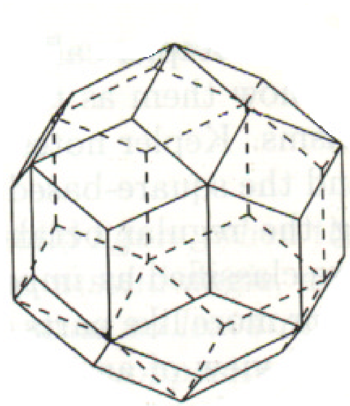


Figure 4.19. A rhombic triacontahedron.

The icosidodecahedron consists of 32 faces of which 20 are triangles and 12 are pentagons. Therefore, its dual, the rhombic triacontahedron consists of 32 vertices of which 20 vertices are surrounded by three rhombic faces and 12 vertices are surrounded by five rhombic faces. The centre polyhedron in Figure 4.20 shows the interchanging of vertices and faces.

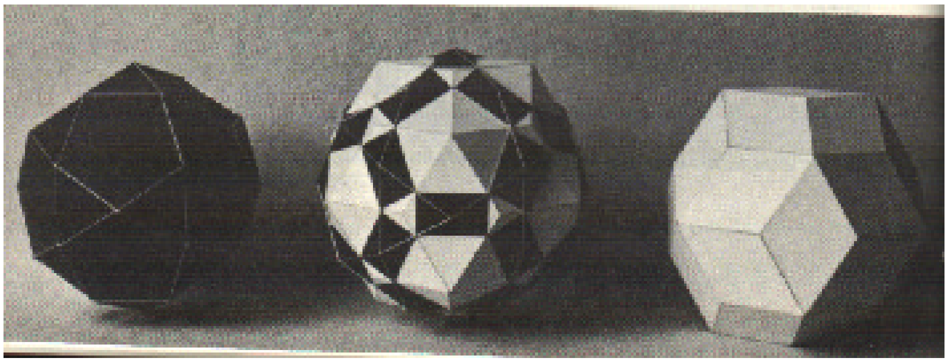
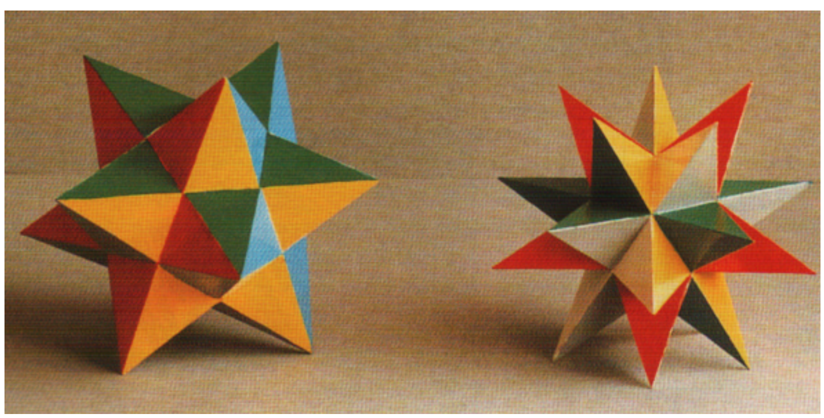


Figure 4.20. The dual pair and its interpenetrating solid.

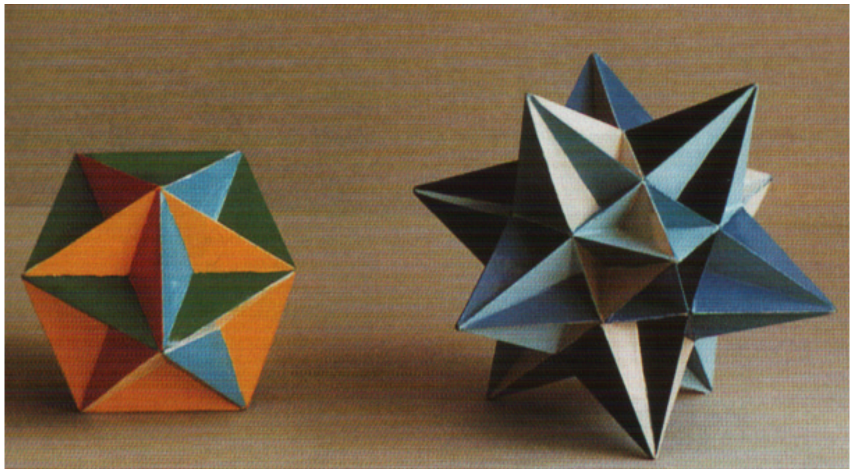
5. Kepler-Poinsot solids

5.1 Introduction

There are four Kepler-Poinsot solids, namely the small stellated dodecahedron, the great stellated dodecahedron, the great dodecahedron and the great icosahedron. All of which are shown in Figure 5.1.



Small stellated dodecahedron   Great stellated dodecahedron



Great dodecahedron Great icosahedron

Figure 5.1. Shows the Kepler-Poinsot solids.

The two with pentagram faces-the small stellated dodecahedron and the great stellated dodecahedron-were described by Kepler in 1619. Poinsot described the other two, the great dodecahedron with regular pentagon faces and the great icosahedron with equilateral triangular faces in 1809. The Kepler-Poinsot solids are regular polyhedra for their faces are regular polygons and the same number of faces surrounds each solid angle. Thus, together with the five Platonic solids they bring the total number of regular polyhedra up to nine. However, unlike Platonic solids, the Kepler-Poinsot solids are concave with intersecting facial planes and are obtained by a process known as stellation.

5.2 What is stellation?

There are two methods of stellation for polyhedra.

1. The process of extending the faces of the polyhedra until they re-intersect is called face-stellation.

2. The process of extending the sides of the faces until they re-intersect is called edge-stellation.

For polygons only edge-stellation is possible. In chapter one it is clear that the stellated polygons are obtained by edge-stellation. For example, if we extend the sides of a pentagon, we get a pentagram or a five-pointed star.

Edge-stellation of the tetrahedron, the cube and the octahedron does not produce new polyhedra. This is because the extended edges do not intersect apart from at the original vertices. Face-stellation of the tetrahedron and the cube also does not produce any stellated forms. This is because if we extend their faces they never intersect. The octahedron has eight triangular faces. Extension of the faces forms a triangular pyramid over each face. Thus, the octahedron has a stellated form, namely the stella octangula.

5.2.1 Face-stellation

The process of face-stellation is approached in two different ways, the two-dimensional approach and the three-dimensional approach. Cromwell explained the two approaches in his book "Polyhedra". His explanations are given below.

1. The two-dimensional approach: choose one face-plane and see how the others intersect it. From the information in this one plane, we then deduce possible faces for stellated forms.

2. The three-dimensional approach: the stellations of a polyhedron are thought of as being built up from layers of solid cells or bounded cells. The bounded cells come in layers surrounding the central, core polyhedron. They can be stuck together to form new polyhedra whose faces will lie in the same planes as those of the original polyhedron.

We will examine the two approaches when discussing stellations of the dodecahedron

5.3 Stellations of the dodecahedron

5.3.1 Small stellated dodecahedron

Edge stellation

The small stellated dodecahedron is constructed by applying edge-stellation on the dodecahedron. Consider a pentagonal face of the dodecahedron. Extend the edges of the pentagonal face until they re-intersect to form a new closed polygon. The new closed polygon formed is the pentagram. This can be seen in Figure 5.2.

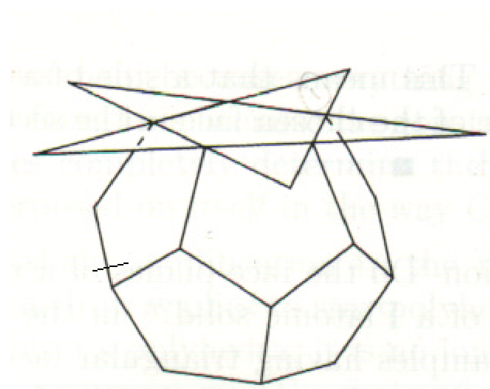


Figure 5.2.

Repeating the process on each face of the dodecahedron forms a pentagonal pyramid above each face. Finally, we will have the small stellated dodecahedron with the dodecahedron contained in it.

Self-intersecting or non-self intersecting?

The small stellated dodecahedron can be regarded as a self-intersecting polyhedron or a non-self intersecting polyhedron.

1. If the small stellated dodecahedron is regarded as a self-intersecting polyhedron, then it consists of 12 intersecting pentagrams with five meeting at each vertex. Edge-stellation on each face of the dodecahedron forms a pentagram. Since the dodecahedron consists of 12 faces, the stellated form has 12 pentagrams. In Figure 5.1 we can see the red, blue, yellow and green pentagram faces intersecting one another. As mentioned earlier, edge-stellation forms a pentagonal pyramid above each face of the dodecahedron. Thus, there are 12 pentagonal pyramids.

   In Figure 5.3 we can clearly see one of the pentagonal pyramids at the centre of one of the pentagram faces. We can also see that five pentagrams meet to form the pyramid. Thus, the small stellated dodecahedron has 12 vertices with five pentagrams meeting at each vertex. Each pentagram has five edges. The small stellated dodecahedron consists of 12 pentagrams with each edge shared by two pentagrams. Thus, the small stellated dodecahedron has (12 x 5)/2 = 30 edges.

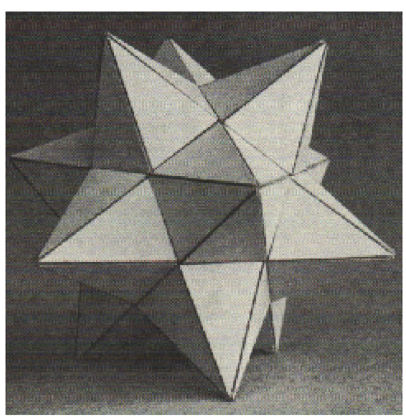


Figure 5.3.

2. If the small stellated dodecahedron is regarded as a non-self intersecting polyhedron then it is better known as the elevated dodecahedron. This is because the small stellated dodecahedron is then regarded as a dodecahedron that has had a pyramid erected on each face. Thus, each pyramid is made up of five isosceles triangles and there are 12 pyramids. Therefore, the elevated dodecahedron consists of 60 isosceles

triangular faces. The dodecahedron has 20 vertices. When a pyramid is elevated on each face, 12 new vertices are formed and they are actually the apexes of the 12 pyramids. Thus, the elevated dodecahedron has 20 + 12 = 32 vertices.

Face stellation

The small stellated dodecahedron is also constructed by face-stellation of the dodecahedron. The following are two ways of looking at face stellation.

1. The two-dimensional approach: if we place a dodecahedron on a table, there is a unique plane parallel to the table containing its top face. The plane containing the base does not meet this top face-plane, but each of the other ten face-planes do. The five planes adjacent to the top face intersect the top face-plane in five lines. These lines bound the top pentagonal face and also form a pentagram. The red pentagram around the blue pentagon in Figure 5.4 represents this. Take all the small acute-angled, red isosceles triangles from each face-plane and the first stellation of the dodecahedron, the small stellated dodecahedron, appears.

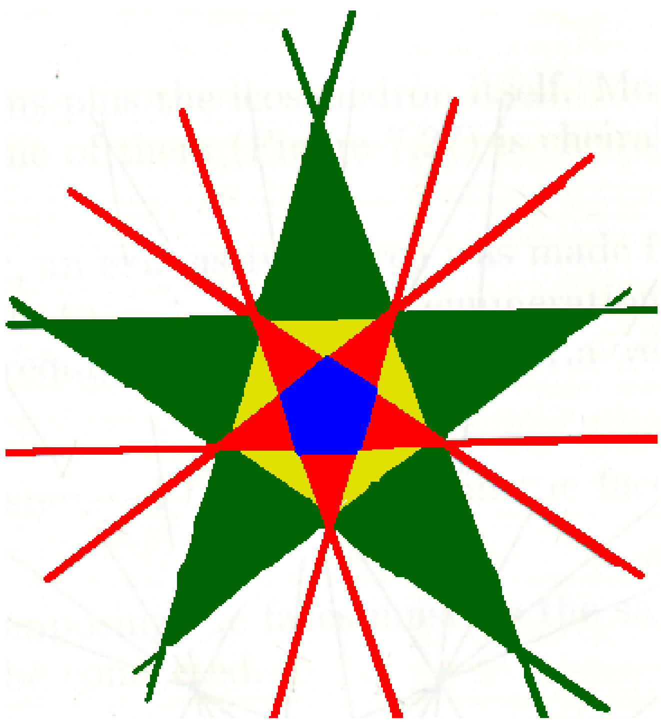


Figure 5.4.

2. The three-dimensional approach: the first stellation of the dodecahedron is built from a layer of 12 pentagonal pyramids. The 12 pentagonal pyramids are the bounded cells with the red triangles (Figure 5.4) for faces. Each pyramid is stuck on a face of the

dodecahedron as seen in Figure 5.5. This is similar to the idea of the elevated dodecahedron mentioned earlier.

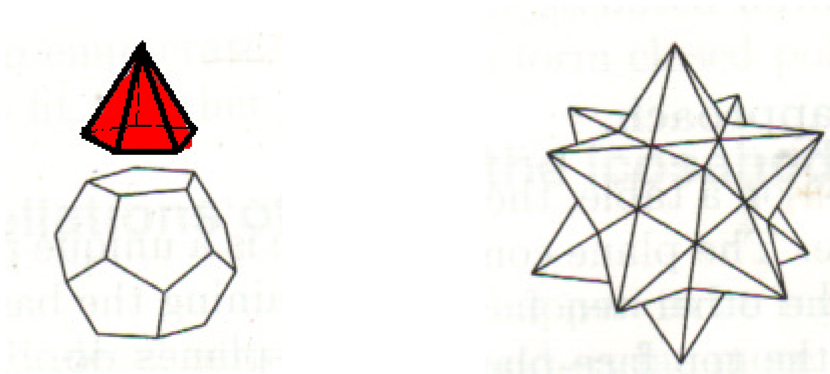


Figure 5.5. The first face stellation of the dodecahedron is built from a layer of 12 pyramids.

5.3.2 Great dodecahedron

The great dodecahedron has 12 self-intersecting pentagonal faces. Also, five pentagons meet at each vertex of the great dodecahedron. The great dodecahedron has 12 vertices and 30 edges. The number of vertices and edges are calculated in the same way as they are calculated for the small stellated dodecahedron.

Face stellation

The great dodecahedron is also constructed by face-stellation of the dodecahedron or the small stellated dodecahedron. The following explain how it is formed using the two approaches of face-stellation.

1. The two-dimensional approach: this involves face-stellation of the dodecahedron to obtain the great dodecahedron. When the five planes containing the faces adjacent to the base are extended to meet the top face-plane, five lines of intersection are produced. The green lines in Figure 5.4 represent the five lines. In Figure 5.4 we can see that the five lines bound a "great face" yellow pentagon and a "great face" green pentagram. The green pentagrams will be explained in the section under the great stellated dodecahedron. The "great face" yellow pentagons form the face of the great dodecahedron.

   The great pentagonal faces obtained from face-stellation of the dodecahedron could be the reason for the name the 'great dodecahedron'. In another way we can also think of the great dodecahedron as being formed from the obtuse- angled, yellow isosceles triangles in Figure 5.4.

2. The three-dimensional approach: this involves face-stellation of the small stellated dodecahedron to obtain the great dodecahedron. The bounded cells in this case are 30 wedges with the yellow isosceles triangles (Figure 5.4) for faces. Each wedge is stuck between the pyramids of the small stellated dodecahedron. This can be seen in Figure 5.6.

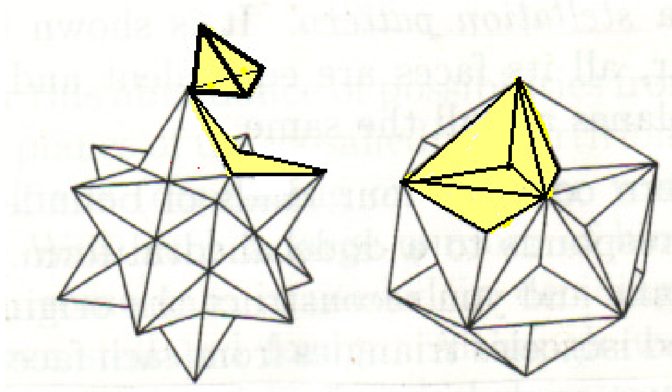


Figure 5.6. The second stellation of the dodecahedron is built from a layer of 30 wedges.

The small stellated dodecahedron is made from 12 pentagonal-based pyramids. Each triangular face of the pyramid is adjacent to a triangular face of an adjacent pyramid. The wedges cover the two adjacent triangular faces from different pyramids. Thus, there are 12 pyramids with five triangles in each pyramid and a wedge covers two triangles. Therefore number of wedges needed is (12 x 5)/2 = 30. Notice how each successive layer of bounded cells completely covers the faces of the previous layer. Since the great dodecahedron is also obtained by face-stellation of the small stellated dodecahedron, it can also be known as the stellated small stellated dodecahedron.

5.3.3 Great stellated dodecahedron

Edge stellation

The great stellated dodecahedron can be constructed from edge-stellation of the great dodecahedron. Extending the edges of each pentagon face of the great dodecahedron forms a pentagram as seen in Figure 5.7. Since the great dodecahedron consists of 12 self-intersecting pentagons, the great stellated dodecahedron consists of 12 self-intersecting pentagrams like the small stellated dodecahedron. However, unlike the small stellated dodecahedron that has five pentagrams at each vertex, the great stellated dodecahedron has only three pentagrams meeting at each vertex. Thus, through similar calculations used earlier, the great stellated dodecahedron has 20 vertices and 30 edges.

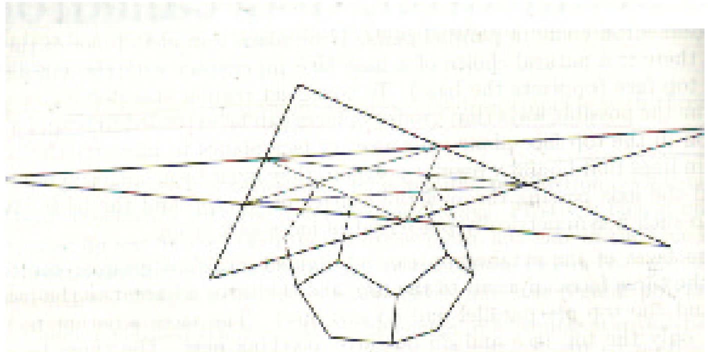


Figure 5.7.

Face stellation

The great stellated dodecahedron is also constructed by face-stellation of the dodecahedron or the great dodecahedron. The following explain how it is formed using the two approaches of face-stellation.

1. The two-dimensional approach: this involves face-stellation of the dodecahedron to obtain the great stellated dodecahedron. Under the two-dimensional approach of the great dodecahedron, the construction of the “great face” green pentagram in Figure 5.4 was mentioned. This green pentagram forms the faces of the great stellated dodecahedron. In another way we can also think of the great stellated dodecahedron as being formed from the big acute-angled, green isosceles triangles in Figure 5.4.

   Both the small stellated dodecahedron and the great stellated dodecahedron are obtained from face-stellation of the dodecahedron. Thus, they have the term ‘stellated dodecahedron’ in their names. Separate face-stellations of the dodecahedron form the red pentagram and the green pentagram. Clearly, the red pentagram is smaller than the green pentagram. The small red pentagram forms the faces of the small stellated dodecahedron and the “great” or large green pentagrams form the faces of the great stellated dodecahedron. This accounts for the ‘small’ and ‘great’ terms used in the names that correspond to the size of the faces of the two stellated dodecahedrons.

2. The three-dimensional approach: this involves face-stellation of the great dodecahedron to obtain the great stellated dodecahedron. The bounded cells in this case are 20 spikes each of which is an asymmetric triangular dipyramid. Figure 5.8 shows a spike. The top pyramid consists of the green triangles (Figure 5.4) for faces. The bottom pyramid of the spike consists of the yellow triangles (Figure 5.4) for faces. Each spike fits into the hollows or depressions between the wedges. The wedges are the bounded cells with yellow faces used to form the great dodecahedron. The wedges create depressions in the great dodecahedron and the depressions are thus made up of yellow triangles (Figure 5.4).

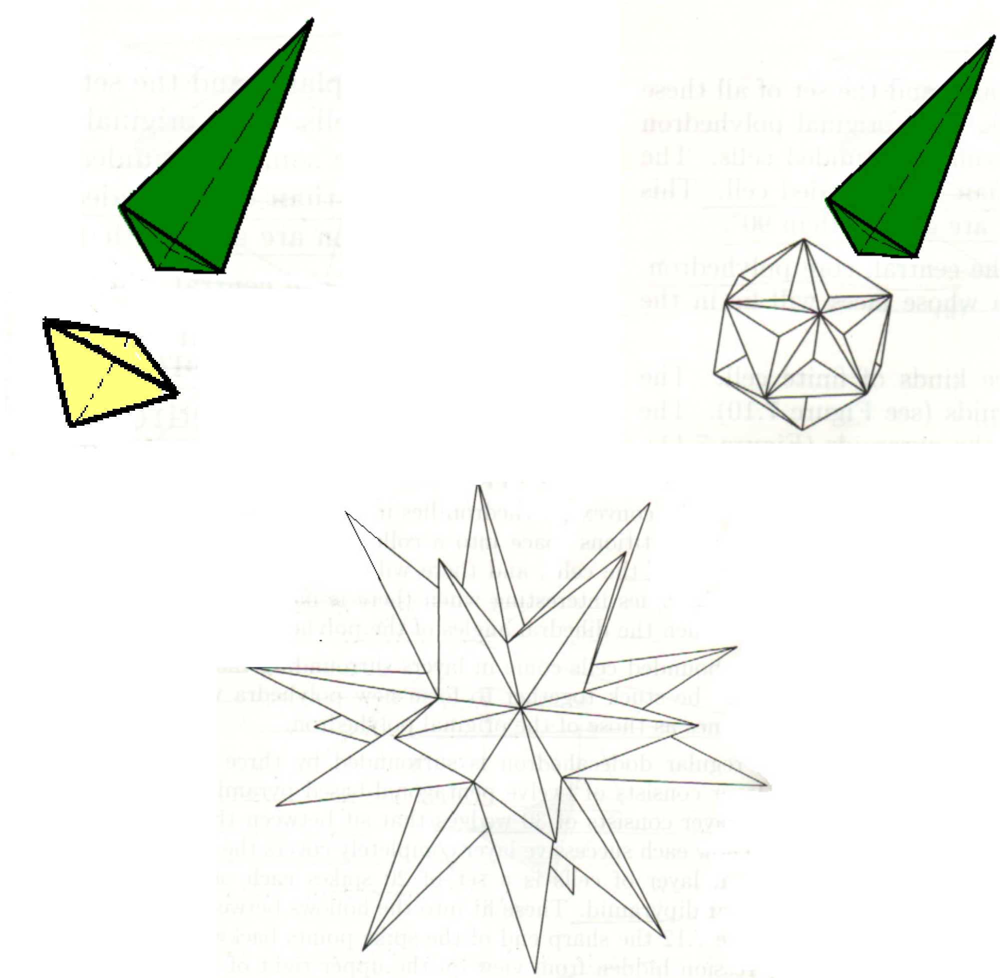


Figure 5.8. The third stellation of the dodecahedron is built from layer of 20 spikes.

Therefore, the bottom pyramid of each spike that is also made up of yellow triangles, fits into the depression hidden from view on the upper right of the great dodecahedron in Figure 5.8. While the top green pyramid (Figure 5.8) containing the sharp end of the spike points backwards. Note that each pentagonal face of the great dodecahedron has five depressions and each depression is shared between three faces. Thus, there are (12 x 5)/3 = 20 depressions. Therefore, 20 spikes are needed. Since the great stellated dodecahedron is also obtained by face-stellation of the great dodecahedron, it can also be known as the stellated great dodecahedron.

Interestingly, the 12 pentagonal faces of the great dodecahedron correspond to the 12 pentagons in the icosahedron. The outline of the five depressions found on each pentagonal face of the great dodecahedron corresponds to the five equilateral triangular faces that meet at each vertex. And the five edges that surround the five triangles at each vertex are the five sides of the pentagon.

Edge-stellation of the icosahedron is similar to edge-stellation of the dodecahedron. Extending the edges of the pentagon in the icosahedron forms a pentagram. Repeating the process for all the pentagons in the icosahedron forms a triangular pyramid on each triangular face of the icosahedron. The triangular pyramid corresponds to the green triangular pyramid (Figure 5.8) because they are both formed from the same regular pentagon. Therefore, the mounting of triangular pyramid on the triangular face of the icosahedron is the same as mounting the green triangular pyramid on the depression of the great dodecahedron. Also, the edge-stellated icosahedron consists of 12 pentagram faces with three pentagrams at each vertex. Therefore, the edge-stellated icosahedron has 20 vertices and 30 edges. This makes the great stellated dodecahedron to be equal to the edge-stellated icosahedron.

However, the great stellated icosahedron is not equal to face-stellated icosahedron. We will explain this under stellations of the icosahedron.

Why are there only three stellations of the dodecahedron?

The ten lines of intersection in Figure 5.4 form a pattern called the stellation pattern. The stellation pattern of the dodecahedron contains three kinds of bounded region, excluding the central pentagon. Each kind of region corresponds to a stellated dodecahedral form as we have seen earlier. The lines in the stellation pattern do not bound any more finite regions and so the great stellated dodecahedron is the last of the dodecahedron stellations.

5.4 Stellations of the Icosahedron

Stellation pattern of the dodecahedron shows that only three stellated forms of the dodecahedron are possible. Figure 5.9 shows the stellation pattern of the icosahedron.

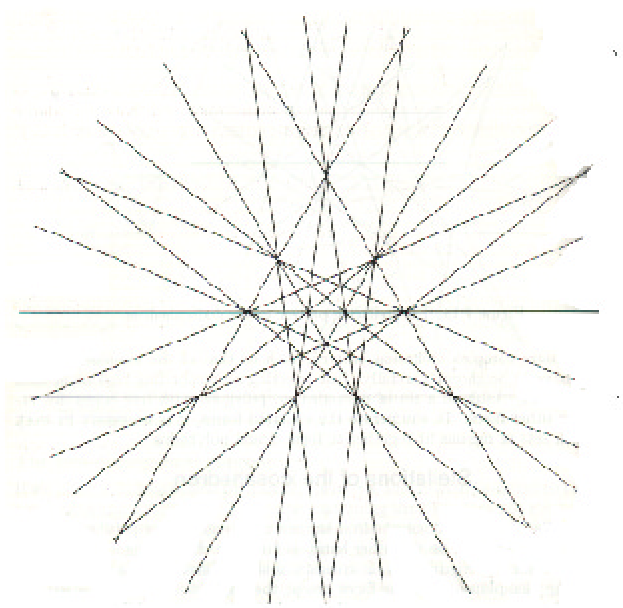


Figure 5.9. Stellation pattern of icosahedron.

It consists of the 18 lines where a face-plane is met by the face-planes of all the faces except the opposite one (which is parallel to it). These lines divide the face-plane into 66 finite regions. However, only certain combinations of the cells formed from the stellated pattern can be considered as stellated polyhedra. The following criteria were proposed by J.C.P. Miller.

1. The faces of the stellated form must lie in the face-planes of the original polyhedron.

2. All the regions composing the faces must be the same in each plane; these regions need not be connected.

3. The regions included in a plane must have the same rotational symmetry as a face of the original polyhedron. Together with condition two, this implies that the stellation process preserves the rotational symmetry of the original polyhedron.

4. The regions included in a plane must be accessible in the completed stellation.

5. Compounds of simpler stellations are excluded. More specifically, we disallow unions of two stellations with no face-to-face contact except combinations of mirror images.

There are 59 face-stellated icosahedra satisfying the above five conditions. Only one of the stellated icosahedron is a regular polyhedron, namely the great icosahedron.

5.4.1 Great Icosahedron

The great icosahedron consists of 20 self-intersecting, equilateral triangular faces as shown in Figure 5.1. Five equilateral triangles meet at each vertex. Thus, the great icosahedron has 12 vertices and 30 edges. Notice that the icosahedron also consists of 20 equilateral triangular faces but they are not intersecting. Also, five equilateral triangles meet at each vertex of the icosahedron. Thus, the icosahedron like the great icosahedron has 12 vertices and 30 edges. The only difference is that the equilateral faces of the great icosahedron are larger than that of the icosahedron. Thus, the stellated icosahedron is called the great icosahedron. This is similar to the reason for the name great dodecahedron. The term 'great' in the names correspond to the size of the faces that are larger than the faces of the corresponding original polyhedra.

While discussing the great stellated dodecahedron, we noted that the edge-stellated icosahedron is the great stellated dodecahedron. However, face-stellated icosahedron is not the great stellated dodecahedron. This is because face stellation of the icosahedron can be made from different triangles obtained from the stellation pattern of the icosahedron. The triangles in the stellation pattern of the icosahedron are not the same as the triangles in the stellation pattern of the dodecahedron. Thus, the green triangles (Figure 5.4) that form the great stellated dodecahedron are not found in face-stellation of the icosahedron. Therefore, face-stellation of the icosahedron is not equal to the great stellated icosahedron.

5.5 Duals of the Kepler-Poinsot solids

Duals of regular polyhedra are regular polyhedra. This is because dualizing involves interchanging of faces and vertices. Each vertex of regular polyhedra is equivalent and has a regular vertex figure. Thus, the faces of the duals consist of one type of regular polygons. The regular polyhedra have one type of regular face. Thus, the duals have equivalent vertices. Remember that dualizing a Platonic solid formed another Platonic solid. Likewise, dualizing a Kepler-Poinsot solid forms another Kepler-Poinsot solid.

The Kepler solids, the small stellated dodecahedron and the great stellated dodecahedron are duals to the Poinsot solids, the great dodecahedron and the great icosahedron, respectively. The Kepler solids have concave faces (pentagrams) and convex corners. The Poinsot solids have convex faces (pentagons and triangles) and concave corners.

6. References:

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