GEM 1518K Mathematics in Art and Architecture

Understanding Escher

A look into the mathematical principles behind his tessellations

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Note: The hard copy version contains drawings that could not be included here

"For me it remains an open question whether [this work] pertains to the realm of mathematics or to that of art."

– M.C. Escher

Introduction

Maurits Cornelis Escher (1898-1972) was schooled in graphic arts and was a prolific creator of enchanting landscapes, spatial illusions and impossible buildings in woodcut, print and lithograph. He is perhaps best known for combining Mathematics with Art in his fascinating tessellations and patterns.

Among his greatest admirers were mathematicians and crystallographers, who recognized in his work an extraordinary visualization of mathematical principles. This was even more remarkable as Escher had no formal mathematical training besides his basic education.

Major themes in his mathematical work revolved around the penetration of worlds (*Three Spheres*), the impossible (*Belvedere, Waterfall*), the infinite (*Circle Limit III*) and of course the focus for this paper: the regular divisions of the plane. As his work developed, he drew great inspiration from the mathematical ideas he read about, and was able to produce increasingly complex works of art.

Snakes

Man with cuboid

During a trip to Moorish Spain, Escher visited the Alhambra, which housed many beautiful and complex tiling patterns. Escher was deeply impressed by the complexity and beauty of the way the Muslims blended geometry and art, and the Muslims were indeed among the most advanced in the world in mathematical art. However, it was forbidden to portray any forms similar to living objects, and so was greatly limited in scope. The challenge Escher embarked on was to build the shapes with recognizable animal forms and meaningful objects, creating his tessellations that are today famous the world over.

Aim

In this paper, we aim to explore the mathematical principles underlying Escher's beautiful and complex tessellations, which seem to defy understanding. We will show how it is possible to break down and simplify some of the common themes in his tessellations and allow the reader to appreciate Escher's work from a whole new perspective and understanding. We also hope to fire the reader's creativity such that he might be able to produce his own tessellations from the methods outlined below.

Escher's sketch from the Alhambra

What are tessellations?

Regular divisions of the plane, called "tessellations," are arrangements of closed shapes that completely cover the plane without overlapping and without leaving gaps. Typically, the shapes making up a tessellation are polygons or similar regular shapes, such as the square tiles often used on floors.

A dictionary will tell you that the word "tessellate" means to form or arrange small squares in a checkered or mosaic pattern. The word "tessellate" is derived from the Ionic version of the Greek word "tesseres," which in English means "four." The first tilings were made from square tiles. In short, a tessellation is any repeating pattern of interlocking shapes. It is any shape that can be put together to cover an entire area with no gaps or holes in between.

Platonic tilings are tessellations formed from regular polygons.

a tiling of triangles
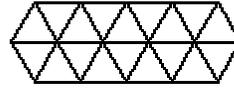
a tiling of squares
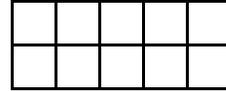
a tiling of hexagons
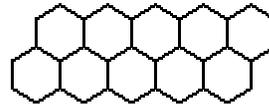
An early
tessellation

Is there a difference between a tessellation and a tiling? Actually, the two words, "tilings" and "tessellations" are often used interchangeably. Tessellations are also sometimes known as tilings, but the word "tilings" usually refers to patterns of polygons (i.e., shapes with straight boundaries), which is a more restrictive category of repeating patterns.

Thus, tessellations differ from tilings in the sense that tessellations can be made of irregular shapes. As we will see, MC Escher made use of a variety of shapes like reptiles and birds as well as Platonic, Archimedean and non-periodic tilings in his patterns. He was also able to skillfully manipulate his tessellations such that they could change into other forms (as shown below).
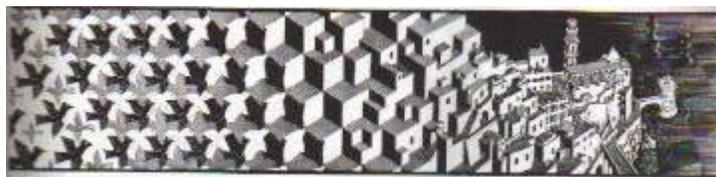

Detail from "Metamorphosis III"

Understanding Escher

To be able to appreciate MC Escher's tessellations from a perspective other than a purely aesthetic one, we can try to apply mathematical principles as it seems that his work is heavily grounded in mathematics. By introducing the concepts of regular divisions of the plane and rigid motions, we can see how his work can be broken down into its simpler components.

In *Regelmatige Vlakverdeling* (The Regular Division of the Plane), Escher explained that "*a plane, which should be considered limitless on all sides, can be filled with or divided into similar geometric figures that border each other on all sides without leaving any 'empty spaces'. This can be carried on to infinity according to a limited number of systems.*"

What Escher was trying to say was that a regular division of the plane was like a jigsaw puzzle with identical pieces. We can easily understand that a pattern of squares lined up edge to edge can fill up a plane *ad infinitum*, with no space in between for gaps. After further examination, we can find that this applies to equilateral triangles and hexagons (the so-called Platonic tilings), and also to a combination of polygons, as illustrated in the various Archimedean and Penrose tilings. However, there is no limit to the types of shapes that can be used: Escher was unique in that he was able to incorporate diverse shapes like angels, birds, and horsemen into his tessellations.

The best way to understand Escher's interlocking shapes is to dissect the figures. You can take a piece of transparency paper and copy the outline of a shape: you will find that by joining the vertices (points where 3 or more neighboring shapes lines touch), you end up with a basic grid pattern similar to that of the platonic tilings discussed before. Escher exploited these basic patterns in his tessellations, applying what geometers would call **translations, reflections**, **rotations** and **glide reflections** to obtain a greater variety of patterns. He also elaborated these patterns by "distorting" the basic shapes to render them into animals, birds, and other figures. These distortions had to obey the three, four, or six-fold symmetry of the underlying pattern in order to preserve the tessellation.
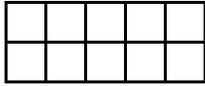
 A division into squares (the 4.4.4.4 tiling)

To be able to understand how he arrived at his tessellations, we will describe the steps he typically takes to create them. In short, Escher used the following steps:

<div style="border:1px solid">

To create a tessellation:

1) **Map** out a regular division of a plane (usually of triangles or hexagons) to get a basic skeleton for the shape
2) **Deform** the basic shapes into familiar figures like that of birds.
3) Apply the respective **rigid motions** to them such that they are able to cover a plane.

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Rigid Motions explained

To understand rigid motions, imagine placing a piece of tracing paper on top of a pattern (eg: a hexagonal platonic tiling). Trace the pattern of hexagons onto the tracing paper such that there is now an original and a duplicate. Next, shift the duplicate around while on top of the original: you will see that the two patterns will not correspond with each other unless moved or rotated in 4 specific ways: **translation, reflection, rotation** and **glide reflection**.

Translation

It is a "shifting" of a shape, where all the shapes are moved in the same direction and the same distance. Shapes are simply translated in a direction without loss of orientation.

Eg:

☆ ☆ ☆ ☆

Reflection

Reflection occurs when an image is "flipped over" along an axis. A way to envisage this is by placing a small mirror along an object to act as an axis of reflection.

Eg:
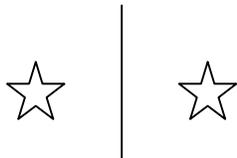
Reflection along vertical axis   Reflection along horizontal axis
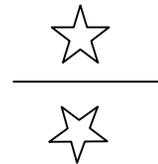
Rotation

To understand rotation, imagine sticking a pin through the duplicate copy of tracing paper and moving it around the pin, which serves as the center of rotation.

Eg:

Centre of rotation
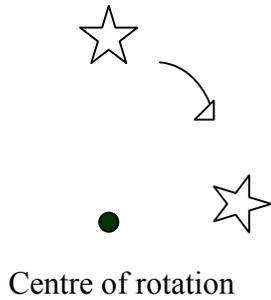
Glide reflection

A glide reflection is a reflection around an axis, combined with a translation along the same axis.

Eg:

Axis of symmetry
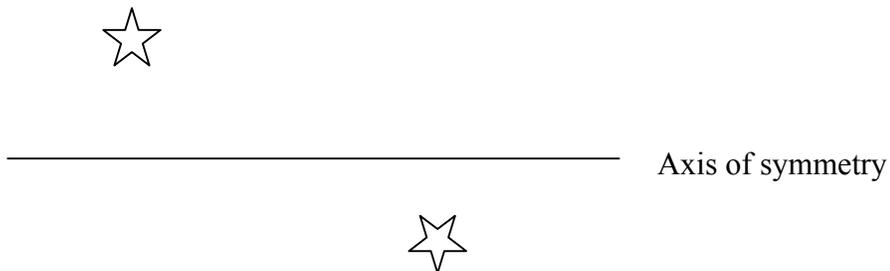
How to create your own simple tessellation:

Start off with a simple division of the plane: here I will use a square to demonstrate.
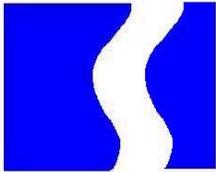
Cut a shape out of any edge.
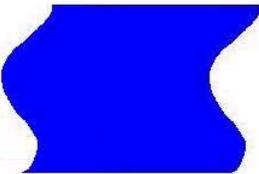
Paste it onto the opposite end. The blue portion that is cut out compensates for the white portion left behind. Thus you cannot cut without leaving a similar shape on either end.
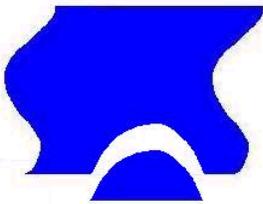
Do the same for below.
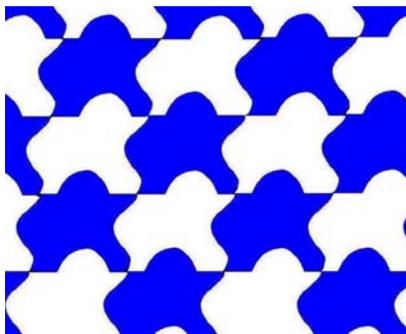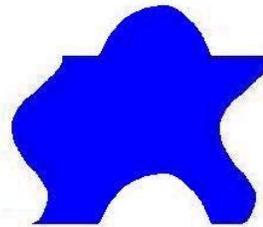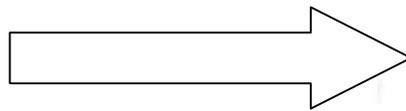
Simply translate the shape and the tessellation is formed.

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Tessellations illustrated: **Plate 29 Translational Symmetry**
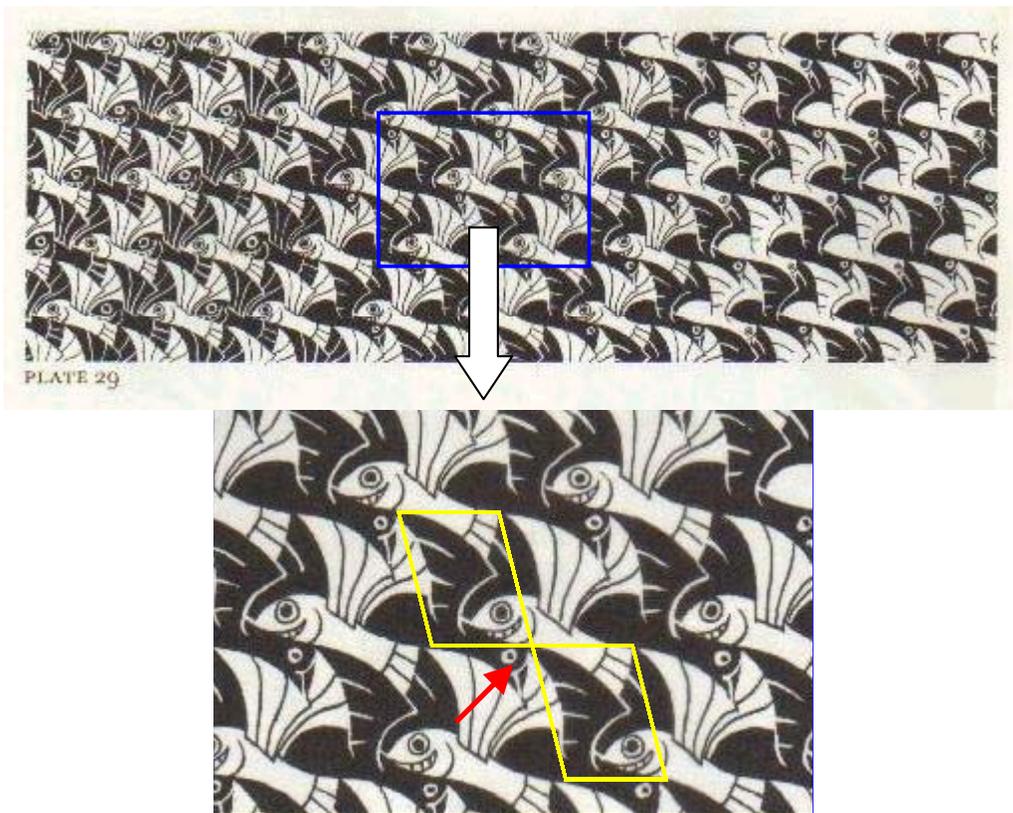
Birds and fishes were a common motif for Escher's tessellations. As you might be able to infer from the yellow diamond, this tessellation is based on a **translation** of a creature made from a rhombus pattern where the yellow rhombus, if pulled over another yellow rhombus, should correspond exactly. Other clues are that 4 figures meet at any given vertex (the red arrow).
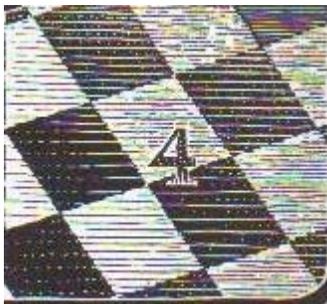
Escher starts off with straight, parallel lines going in two directions to form a checkered pattern from which he fashions his objects.

Using the method of deformation illustrated above, he alters the basic shape of the parallelogram such that it begins to take on a familiar shape: that of a bird or a fish. As you can notice, each protrusion is compensated on the opposite end of the rhombus. Also, the outline of the bird should not exceed the boundaries of the parallelogram grid by too much.

The finished product: Escher then fills in the shapes with details. You can see that he plays on the stark contrast of black and white by pushing the fish into the background whenever the bird is in detail on the right side, and vice versa on the left side. This can also be seen in his famous print "*Day and Night*".

# Tessellations Illustrated: **Plate 37 Rotational Symmetry**
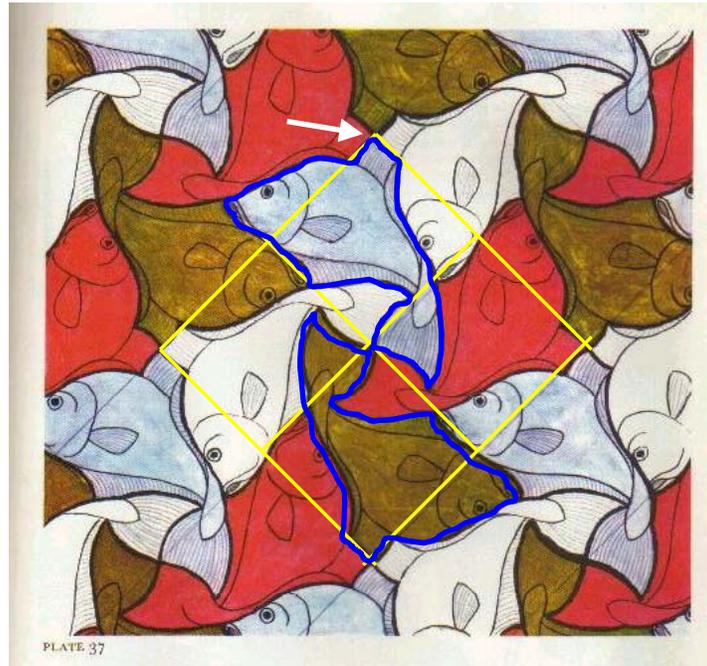
Plate 37 is a good example in which Escher uses the principles of **rotation** to create a beautiful tessellation with interlocking fish. The basic framework of this pattern is that of a square tiling where squares are lined up against each other. We can probably infer this as at any given instance, only four fish meet at various vertices. As you can see, there is a four-fold rotational symmetry where the blue outline of the fish can be rotated 90°, 180°, 270° and 360° around the vertex in the middle of the four yellow squares where four tails meet. Of course, this is not the only possible centre of rotation: the white arrow points to another centre where four fins meet.
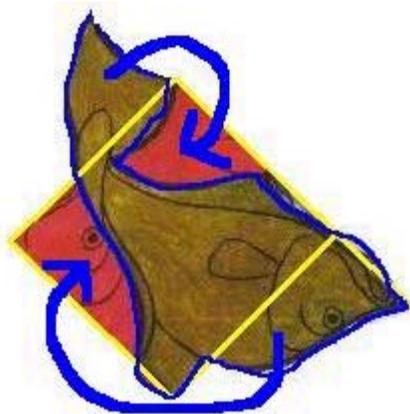
Tessellations Illustrated: **Plate 17 Glide Reflection**
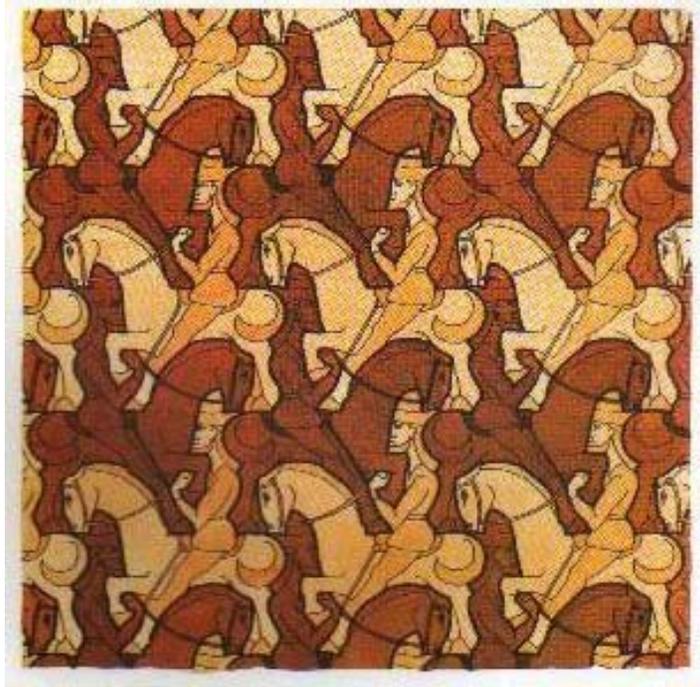
This is the famous "Horseman" pattern that was featured in scientific books detailing the derivation of crystal symmetries and also in a book on elementary particles. The pattern seems to be built on a translation but that is only applicable to the white horses or the brown horses if taken alone, and not to both. Upon closer inspection, we found that the brown horses were actually a mirror **reflection** of the white horses, and this takes place along a vertical axis. This combines to form a **glide reflection**.
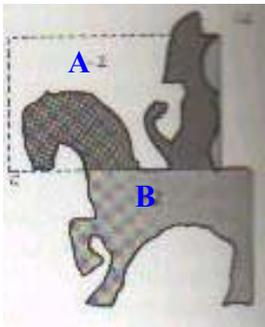
The shape of the horseman can be derived from a simple rectangle. Escher starts with the upper rectangle, cutting out Shape A. He then flips the shape through a vertical reflection and he translates it to become Shape B. This creates the pattern of the horseman, which can be used to fill the plane.

A) The horseman on the right side goes through a reflection about the vertical white line (so it has vertical reflective symmetry).

B) The reflected horseman is then translated upwards. A repetition of this across the plane would lead to the Horseman tessellation being formed.

More tips on creating tessellations

Escher said that he felt irresistible joy with putting multiple copies of an image on a drawing and making them fit together in a structure. How can you replicate this joy?

Over a period of 40 years, Escher gradually learnt about the mathematical rules and about techniques used by other cultures, and the science of packing molecules. He found that within each interlocking shape, half of the boundary is a clone of the other half. These identical sections of the boundary make it possible for neighboring shapes to mate together without gaps. This pre-condition often prevents an arbitrary shape from mating together, or it makes a good fitting shape look meaningless (an absurd silhouette).
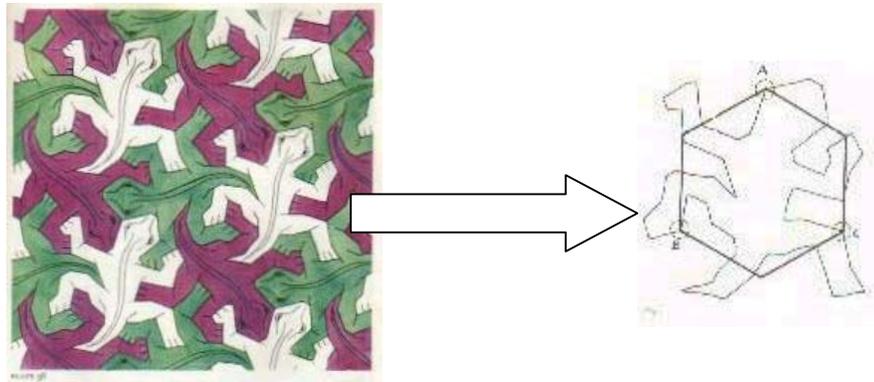
Another general rule is that the boundary of a silhouette (for example, the hexagon above) should not be crossed too much by the shape (the lizard). The vertices of Escher's lizard drawing form a regular hexagon. Three of the vertices are rotation points for the 3 sides of the lizard. It is difficult to let these lines come to close proximity (to form the legs of the lizard) and yet never intersect another boundary line, and thus takes quite some trial and error. If you notice, the head of the lizard is cut out of the part just to the left of Vertex A, and so on for the other parts.

Escher himself gave us some tips in *Regelmatige Vlakverdeling*.

Firstly, a shape to be formed into a tessellation should be of an enclosed form, which encloses the whole figure with an unbroken outline. What he means is that the repertoire of objects we can tessellate easily is limited because objects like plants, which are connected to the ground more obviously than animals, are difficult to depict "without giving an indication of what they are joined to".
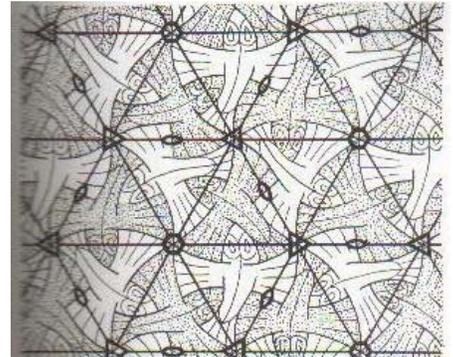
A tessellation of birds with six-fold symmetry. Birds are a common theme they possess the "necessary angularity".

Secondly, the outline should be as characteristic and distinct as possible, with little ambiguity. He explains that "the effect of the silhouette should be so striking that the object can be recognised even without any internal detail, which always has a disturbing effect on the figure".

Thirdly, he stresses the necessity of the figure having no "excessively deep or shallow indentations or bulges in the outline", because they make the figure difficult to comprehend.

Some objects are also inherently suitable. Escher explains the recurring theme of birds and fish where the shape of a bird has "the necessary angularity", with well-balanced protrusions that are easily incorporated into the body. Also, it has a "characteristic shape", as with fish.

Conclusion

We hope that our project was able to highlight the basic mathematical principles found in Escher's tessellations clearly, and to serve as a starting point from which the reader is able to create his own unique creations and experience the same joy he did when making his works of art.

Works referenced:

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