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Regular Polytopic Distances

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Abstract. Let M be an n-dimensional regular polytope of simplices, hypercubes, or orthoplexes and r be the circumscribed radius of M. If q^4 is the average of fourth powers of distances between a point and vertices of M and s^2 is the average of squares of those distances, then

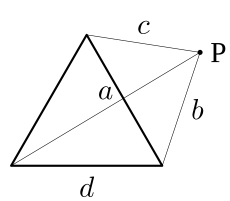
$$q^4 + \frac{4(n+1)}{n^2} r^4 = \left(s^2 + \frac{2}{n} r^2\right)^2.$$

1. Introduction

In his book *Mathematical Circus*, Martin Gardner posed a beautifully symmetric formula satisfied by distances between an arbitrary point and vertices of an equilateral triangle. That is, if a, b, and c are the distances between a point P and three vertices of an equilateral triangle of side d, the relation

$$3(a^4 + b^4 + c^4 + d^4) = (a^2 + b^2 + c^2 + d^2)^2$$

holds (Figure 1).



$$3(a^4 + b^4 + c^4 + d^4)$$
$$= (a^2 + b^2 + c^2 + d^2)^2$$

Figure 1. Equilateral triangle

This result was generalized in two ways by J. Bentin [1, 2]. One is about $(n-1)$-dimensional simplices of side d_0. In this case, if the n distances are denoted by $d_1, d_2, \dots, d_n$, then

$$n(d_0^4 + d_1^4 + \cdots + d_n^4) = (d_0^2 + d_1^2 + \cdots + d_n^2)^2$$

holds (Figure 2).

Another way of Bentin's generalizations is about regular polygons. If we denote the average of fourth powers of distances by q^4 and the average of squares of distances by s^2, then

$$q^4 + 3r^4 = (s^2 + r^2)^2$$

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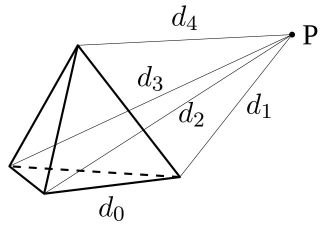
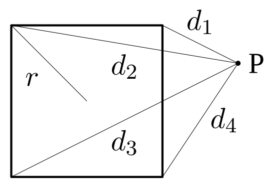


$$4(d_0^4 + d_1^4 + d_2^4 + d_3^4 + d_4^4) \\ = (d_0^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2)^2$$

Figure 2. Regular tetraheron

for arbitrary regular polygon of circumscribed radius r (Figure 3). This formula for equilateral triangles coincides with the above one introduced by Gardner.



$$q^4 + 3r^4 \\ = (s^2 + r^2)^2$$

Figure 3. Square

It is natural to ask whether similar formulas might be obtained for other regular polytopes. We can find ones for cubes and octahedrons. Besides, the results are extended to the higher dimension, that is, hypercubes and orthoplexes.

2. Main results

It is well known that there are only five regular polyhedrons. In dimension 4 there are 6 kinds of regular polytopes. But, dimension 5 or higher allows only three kinds of regular polytopes: n-simplex, n-cube, n-orthoplex. These regular polytopes are denoted by using Schläfli symbols (see [3]).

$$\begin{aligned}
n = 2:&\quad \{k\}, \text{where } k \geq 3 \text{ is an arbitrary integer} \\
n = 3:&\quad \{3,3\}, \{3,4\}, \{4,3\}, \{3,5\}, \{5,3\} \\
n = 4:&\quad \{3,3,3\}, \{3,3,4\}, \{4,3,3\}, \{3,4,3\}, \{3,3,5\}, \{5,3,3\} \\
n \geq 5:&\quad \{3^{d-1}\}, \{3^{d-2}, 4\}, \{4, 3^{d-2}\}
\end{aligned}$$

We find distance relations for n-simplex, n-cube, n-orthoplex with $n \geq 2$. Note that 2-simplex is an eqilateral triagle, 2-cube is a square, and 2-orthogonal is also a square.

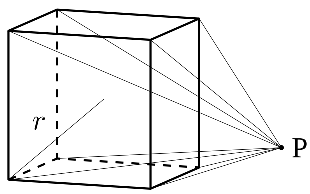
Theorem 1. *For n-dimensional regular simplices, let q^4 be the average of fourth power of distances between $n+1$ vertices and a point and s^2 be the average of squares of those distances. If r is the circumscribed radius, then*

$$q^4 + \frac{4(n+1)}{n^2} r^4 = \left(s^2 + \frac{2}{n} r^2\right)^2.$$

Proof. Just restatement of Bentin's. □

Theorem 2. *For n-dimensional cubes, let q^4 be the average of fourth power of distances between 2^n vertices and a point and s^2 be the average of squares of those distances. If r is the circumscribed radius, then*

$$q^4 + \frac{4(n+1)}{n^2} r^4 = \left(s^2 + \frac{2}{n} r^2\right)^2 .$$



$$q^4 + \frac{16}{9} r^4 = \left(s^2 + \frac{2}{3} r^2\right)^2$$

Figure 4. Cube

Proof. The number of vertices of an n-dimensional cube is 2^n. We may assume that the vertices are represented by $\pm a\mathbf{e}_1 \pm a\mathbf{e}_2 \pm \cdots \pm a\mathbf{e}_n$ in $\mathbb{R}^n$, where $\mathbf{e}_i$ is the elementary unit vector. Then the circumscribed radius is $r = \sqrt{n}a$.

Each vertex is determined by the sequence of 1 or -1. Using functions $\sigma_i : \{1, 2, \ldots, n\} \to \{1, -1\}$ for $i = 1, \ldots, 2^n$, we may write each vertex V_i as

$$(\sigma_i(1)a, \sigma_i(2)a, \ldots, \sigma_i(n)a) .$$

Let $P = (x_1, \ldots, x_n)$ and $\ell^2 = x_1^2 + \cdots + x_n^2$. Then,

$$\begin{aligned} d_i^2 = \|P - V_i\|^2 &= \sum_{k=1}^{n} (x_k - \sigma_i(k)a)^2 \\ &= \sum_{k=1}^{n} \left(x_k^2 - 2x_k\sigma_i(k)a + a^2\right) \\ &= \ell^2 + na^2 - 2a \sum_{k=1}^{n} \sigma_i(k)x_k \end{aligned}$$

Summing up all d_i^2, we obtain

$$\begin{aligned} 2^n s^2 &= d_1^2 + \cdots + d_{2^n}^2 \\ &= \sum_{i=1}^{2^n} \left(\ell^2 + na^2 - 2a \sum_{k=1}^{n} \sigma_i(k)x_k\right) \\ &= 2^n(\ell^2 + na^2) - 2a \sum_{k=1}^{n} \sum_{i=1}^{2^n} \sigma_i(k)x_k. \end{aligned}$$

Since $\sum_{i=1}^{2^n} \sigma_i(k) = 0$,

$$ns^2 + 2r^2 = n(\ell^2 + na^2) + 2na^2 = n(\ell^2 + (n+2)a^2).$$

Now, consider d_i^4. Note that

$$\begin{aligned} d_i^4 &= \left(\ell^2 + na^2 - 2a\sum_{k=1}^{n}\sigma_i(k)x_k\right)^2 \\ &= \ell^4 + n^2a^4 + 4a^2\left(\sum_{k=1}^{n}\sigma_i(k)x_k\right)^2 \\ &\quad + 2\ell^2na^2 - 4na^3\sum_{k=1}^{n}\sigma_i(k)x_k - 4\ell^2a\sum_{k=1}^{n}\sigma_i(k)x_k. \end{aligned}$$

Then,

$$\begin{aligned} 2^nq^4 &= d_1^4 + \cdots + d_{2^n}^4 \\ &= 2^n\left(\ell^4 + n^2a^4 + 2\ell^2na^2\right) \\ &\quad + 4a^2\sum_{i=1}^{2^n}\left(\sum_{k=1}^{n}\sigma_i(k)x_k\right)^2 - (4na^3 + 4\ell^2a)\sum_{i=1}^{2^n}\sum_{k=1}^{n}\sigma_i(k)x_k. \end{aligned}$$

Since

$$\left(\sum_{k=1}^{n}\sigma_i(k)x_k\right)^2 = \sum_{k=1}^{n}x_k^2 + \sum_{k\neq j}\sigma_i(k)\sigma_i(j)x_kx_j$$

and

$$\sum_{i=1}^{2^n}\sigma_i(k)\sigma_i(j) = 0 \text{ with } j \neq k,$$

we obtain

$$2^nq^4 = 2^n(\ell^4 + n^2a^4 + 2\ell^2na^2 + 4a^2\ell^2)$$

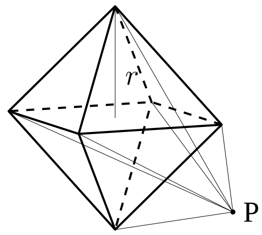
and thus

$$\begin{aligned} n^2q^4 + 4(n+1)r^4 &= n^2(\ell^4 + n^2a^4 + 2\ell^2na^2 + 4a^2\ell^2) + 4(n+1)n^2a^4 \\ &= n^2(\ell^4 + n^2a^4 + 2\ell^2na^2 + 4a^2\ell^2 + 4(n+1)a^4) \\ &= n^2(\ell^4 + 2(n+2)\ell^2a^2 + (n+2)^2a^4) \\ &= (ns^2 + 2r^2)^2, \end{aligned}$$

which is the required result. □

Theorem 3. *For n-dimensional orthoplexes, let q^4 be the average of fourth power of distances between $2n$ vertices and a point and s^2 be the average of squares of those distances. If r is the circumscribed radius, then*

$$q^4 + \frac{4(n+1)}{n^2}r^4 = \left(s^2 + \frac{2}{n}r^2\right)^2.$$



$$q^4 + \frac{16}{9}r^4 = \left(s^2 + \frac{2}{3}r^2\right)^2$$

Figure 5. Octahedron

Proof. An n-dimensional orthoplex has $2n$ vertices. We may assume that the vertices are represented by $V_{i,+} = a\mathbf{e}_i$ and $V_{i,-} = -a\mathbf{e}_i$ for $i = 1, 2, \ldots, n$. Then the circumscribed radius is $r = a$.

Let $P = (x_1, \ldots, x_n)$ and $\ell^2 = x_1^2 + \cdots + x_n^2$. Then,

$$\begin{aligned} d_{i,+}^2 &= \|P - V_{i,+}\|^2 \\ &= x_1^2 + \cdots + x_{i-1}^2 + (x_i - a)^2 + x_{i+1}^2 + \cdots + x_n^2 \\ &= x_1^2 + \cdots + x_{i-1}^2 + x_i^2 + x_{i+1}^2 + \cdots + x_n^2 - 2x_i a + a^2 \\ &= \ell^2 - 2x_i a + a^2 \end{aligned}$$

and

$$\begin{aligned} d_{i,-}^2 &= \|P - V_{i,-}\|^2 \\ &= x_1^2 + \cdots + x_{i-1}^2 + (x_i + a)^2 + x_{i+1}^2 + \cdots + x_n^2 \\ &= \ell^2 + 2x_i a + a^2 \end{aligned}$$

Thus,

$$2ns^2 = \sum_{i=1}^{n} (d_{i,+}^2 + d_{i,-}^2) = 2n\ell^2 + 2na^2$$

and

$$ns^2 + 2r^2 = n\ell^2 + na^2 + 2a^2 = n\ell^2 + (n+2)a^2.$$

Since

$$\begin{aligned} d_{i,+}^4 &= \left(\ell^2 - 2x_i a + a^2\right)^2 \\ &= \ell^4 + 4x_i^2 a^2 + a^4 - 4\ell^2 x_i a - 4x_i a^3 + 2\ell^2 a^2 \end{aligned}$$

and

$$\begin{aligned} d_{i,-}^4 &= \left(\ell^2 + 2x_i a + a^2\right)^2 \\ &= \ell^4 + 4x_i^2 a^2 + a^4 + 4\ell^2 x_i a + 4x_i a^3 + 2\ell^2 a^2, \end{aligned}$$

we obtain

$$\begin{aligned}2nq^4 &= \sum_{i=1}^{n}(d_{i,+}^4 + d_{i,-}^4)\\ &= 2n\ell^4 + 8(x_1^2 + \cdots + x_n^2)a^2 + 2na^4 + 4n\ell^2a^2\\ &= 2n\ell^4 + 8\ell^2a^2 + 2na^4 + 4n\ell^2a^2.\end{aligned}$$

Thus,

$$\begin{aligned}n^2q^4 + 4(n+1)r^4 &= n^2\ell^4 + 4n\ell^2a^2 + n^2a^4 + 2n^2\ell^2a^2 + 4(n+1)a^4\\ &= n^2\ell^4 + 2n\ell^2(n+2)a^2 + (n+2)^2a^4\\ &= (ns^2 + 2r^2)^2.\end{aligned}$$

We are done. □

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[1] J. Bentin, Regular simplicial distances, *Math. Gaz.* **79**(March 1995), p. 106.
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[3] H.S.M.Coxeter, *Regular Polytopes*, Dover Publications Inc., New York 1973.

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