The chaotic obliquity of the planets

J. Laskar & P. Robutel

Astronomie et Systèmes Dynamiques, Bureau des Longitudes, 77 Avenue Denfert-Rochereau, F-75014 Paris, France

Numerical study of the global stability of the spin-axis orientation (obliquity) of the planets against secular orbital perturbations shows that all of the terrestrial planets could have experienced large, chaotic variations in obliquity at some time in the past. The obliquity of Mars is still in a large chaotic region, ranging from 0° to 60°. Mercury and Venus have been stabilized by tidal dissipation, and the Earth may have been stabilized by capture of the Moon. None of the obliquities of the terrestrial planets can therefore be considered as primordial.

THE problem of the origin of the obliquities of the planets (that is, the orientation of their spin axis) is important, because if the obliquities are primordial, they could provide some dynamical constraints on the formation of the Solar System[1-4]. We have investigated how long-term perturbations by other planets affect the obliquities and precession rates of all major planets of the Solar System, and demonstrate that none of the inner planets (Mercury, Venus, the Earth and Mars) can be considered to have primordial obliquities. Any of these planets could have started with nearly zero obliquity, in a prograde state, and the chaotic behaviour of their obliquity could have driven them to their current value. If their primordial spin rate was high enough, Mercury and Venus could have undergone large and chaotic changes of obliquity from 0° to ~90°, before dissipative effects ultimately drove them to their current obliquity. The present obliquity of the Earth may have been reached during a chaotic state before the capture of the Moon[5]. The obliquity of Mars is chaotic at present, with possible variations from 0° to ~60°. The obliquities of the outer planets are stable, but may have gone through a similar process in an earlier stage of the Solar System's formation.

Methods

If the motion of the Solar System were quasiperiodic, the motion of precession could be modelled by equations (3) and (5) in Box 1. One can then use the argument of resonance overlap[6] to suggest whether the motion of precession will become chaotic, or will stay regular. This can be done for the outer planets, but to obtain more accurate results, we integrated directly the full equations of precession (equations (3) and (4)), and did a frequency analysis (Box 2) of the precession solution. As the dynamical ellipticity $(C-A)/C$ is proportional[7] to ν^2, the precession constant α is proportional to the spin rate of the planet ν. For a given value of α, we numerically integrated the precession equations, for all values of the initial obliquity, in steps of 0.1°. These integrations use the orbital solution La90 (ref. 8) as an input and are, in general, done over 18 Myr with a step size of ~200 yr. We then used frequency analysis to obtain a frequency curve whose regularity tells us about the regularity of the precession and obliquity. We also recorded the maximum and minimum value reached by the obliquity during these integrations (Figs 3, 4).

The primordial spin rates of Mercury, Venus and the Earth are not known precisely, so for these planets, we did the above analysis for a wide range of α to obtain a general view of how the obliquity changed with time (Fig. 5). In Fig. 5, regular motion is represented by small dots, while the chaotic zones are larger black points. The chaotic zones would be even larger if we took into account the chaotic diffusion of the orbital solution over timescales of a billion years[8].

Past and present obliquities

Mercury. Mercury's present spin is very low, and apparently trapped in a (2:3) spin-orbit resonance. Most probably, the

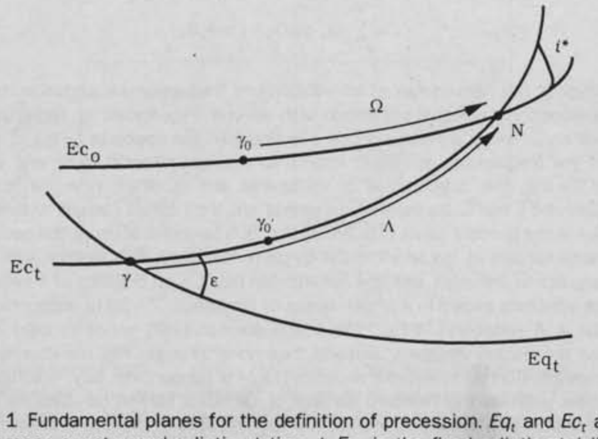

FIG. 1 Fundamental planes for the definition of precession. Eq_t and Ec_t are the mean equator and ecliptic at time t. Ec_0 is the fixed ecliptic at Julian date 2000, with equinox γ_0. The general precession in longitude ψ is defined by $\psi=\Lambda-\Omega$. γ_0' is defined by $\gamma_0'N=\gamma_0N=\Omega$; i^* is the inclination.

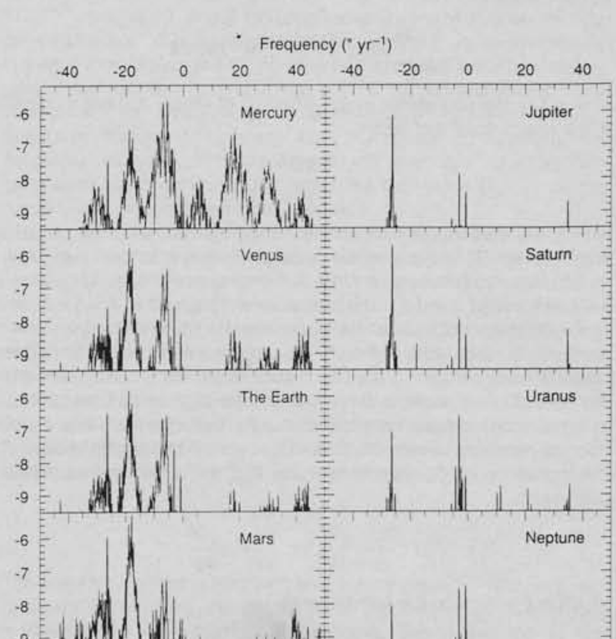

FIG. 2 Fourier spectrum (with Hanning filter) of the planetary forcing term in inclination $A(t)+iB(t)$. The logarithm of the amplitude of the Fourier coefficients is plotted against the frequency (" yr^{-1}).

primordial spin rate of Mercury was much higher and was slowed down by solar tides. Extrapolating the apparent correlation of angular momentum with mass[9], the primordial rotation period of Mercury is found to be 19 h (ref. 10), indicating a precession frequency of $\sim 127'' \text{yr}^{-1}$. Tidal dissipation and core-mantle interactions will cause this frequency to decrease[11,12,32]. For a rotation period smaller than 100 h, there still exists some regular motion for small obliquities, but when the spin rate of the planet decreases, for $\alpha \approx 20'' \text{yr}^{-1}$, the obliquity enters a very large chaotic zone, extending from 0° to ~100° (Figs 3, 5*a*). During this period, and until tidal dissipation ultimately drives Mercury into its present state, the orientation of Mercury undergoes very large and chaotic changes from 0° to 100° within a few million years. For example, for $\alpha = 10'' \text{yr}^{-1}$ (Fig. 3), the obliquity varies from 0 to more than 90° within a few million years for any value of the initial obliquity within this range.

Provided its primordial rotation period was smaller than 300 h (Fig. 5), Mercury must therefore have suffered large-scale chaotic behaviour during its history. At some point its orientation must have been very different from at present (with the pole facing the Sun) and this could have left some traces on the planet.

Venus. The orientation of the pole of Venus (178°) is one of the puzzling features of the Solar System[1,13,14]. Dissipative effects of core-mantle interactions and atmospheric tides can account for some change in spin orientation, but apparently only from 90° to the present value[13,14]. It is thus generally assumed that the retrograde rotation of Venus is primordial, which is one of the arguments supporting the stochastic acretion mechanism for the terrestrial planets[3,4].

As for Mercury, we investigated the global dynamics of the precession and obliquity of Venus using frequency analysis. According to Goldreich[10], the primordial rotation period of Venus was ~13 h, which gives a precession constant of $31'' \text{yr}^{-1}$. We started our investigation with a frequency constant of $40'' \text{yr}^{-1}$, which reveals a regular behaviour for small obliquities, and then a large chaotic region from 50° to 90°. Tidal dissipation due to the Sun will decrease the planet's rotation speed, and consequently its precession constant. When the precession constant reaches $20'' \text{yr}^{-1}$, corresponding to a period of ~20 h, the chaotic zone extends from 0° to nearly 90° (Fig. 3). In Fig. 3, we can see several well defined chaotic zones, but the frequency analysis shows that no invariant surface bounds the motion, so the orbits can diffuse in a few million years from 0° to nearly 90°. As the precession frequency continues to decrease, the shape

BOX 1 Precession and obliquity

Let $A = B < C$ be the principal moments of inertia of a planet. We assume that the axis of rotation of the planet is also its axis of maximum momentum of inertia C. The spin angular momentum is $\mathbf{H} = H\hat{\mathbf{H}}$, with $\hat{\mathbf{H}}$ a unit vector, and $H = Km\mathcal{R}^2\nu = C\nu$, where K is a coefficient depending on the internal structure of the planet ($K = 2/5$ for an homogeneous sphere), $\mathcal{R}$ its equatorial radius, m its mass, and ν its spin rate. Let $\hat{\mathbf{r}}$ be the unit vector in the direction of the Sun, and r the distance to the Sun. The torque exerted by the Sun, limited to first order in $\mathcal{R}/r$, is

$$\mathbf{L} = \frac{3GM}{r^3}\hat{\mathbf{r}} \times I\hat{\mathbf{r}} \quad (1)$$

where I is the matrix of inertia[23,24], G the gravitational constant, and M the solar mass. The precession motion of the planet is given by $d\mathbf{H}/dt = \mathbf{L}$. By averaging over the time (and thus over the mean anomaly), we obtain the equations of precession; that is, the secular motion of the spin axis of the planet

$$\frac{d\hat{\mathbf{H}}}{dt} = \alpha(1-e^2)^{-3/2}(\hat{\mathbf{H}} \cdot \hat{\mathbf{Z}})\hat{\mathbf{H}} \times \hat{\mathbf{Z}} \quad (2)$$

where $\hat{\mathbf{Z}}$ is the unit vector in the direction of orbital angular momentum, e the eccentricity, and where

$$\alpha = \frac{3GM}{2a^3\nu}\frac{C-A}{C}$$

with a the semi-major axis of the planet, will be called the precession constant. Let E_{c0} be the orbital plane at the origin of time with equinox γ_0, E_{ct} the orbital plane at time t, with equinox γ (Fig. 1), and N the intersection of E_{c0} and E_{ct}. The precession in longitude is $\psi = \Lambda - \Omega$, where $\Omega = \gamma_0 N$ is the longitude of the node, and $\Lambda = \gamma N$. We also let γ_0' be the point on E_{ct} such that $\gamma_0' N = \Omega$. The coordinates of $\hat{\mathbf{H}}$ in the reference frame E_{ct} with origin γ_0' are $(\sin\psi \sin\varepsilon, \cos\psi \sin\varepsilon, \cos\varepsilon)$, where ε is the obliquity (the angle between the Equator (E_{qt}) and orbital plane (E_{ct}) at time t). After some calculus, and using the action variable $X = \cos\varepsilon$ and the notations $p = \sin(i^*/2)\sin(\Omega)$, $q = \sin(i^*/2)\cos(\Omega)$ where i^* is the inclination of E_{ct} with respect to E_{c0}, we obtain the equations of precession

$$\frac{d\psi}{dt} = \frac{\partial\mathcal{H}}{\partial X}; \quad \frac{dX}{dt} = -\frac{\partial\mathcal{H}}{\partial\psi} \quad (3)$$

which are related to the hamiltonian

$$\mathcal{H}(X, \psi, t) = \frac{\alpha}{2}(1-e(t)^2)^{-3/2}X^2 + \sqrt{1-X^2}(A(t)\sin\psi + B(t)\cos\psi) \quad (4)$$

with

$$A(t) = 2(\dot{q} + p(q\dot{p} - p\dot{q}))/\sqrt{1-p^2-q^2}$$
$$B(t) = 2(\dot{p} - q(q\dot{p} - p\dot{q}))/\sqrt{1-p^2-q^2}$$
$$C(t) = (q\dot{p} - p\dot{q})$$

The expression $A(t) + iB(t)$ and the eccentricity $e(t)$ describe the orbital motion of the planet and will be given by the La90 solution of Laskar[8]. As the orbital motion of the planets, and especially of the inner planets, is chaotic[8,18,25-27], a quasiperiodic approximation of $A(t) + iB(t)$ is not well suited for obtaining an accurate solution over a few million years. This is reflected in the filtered Fourier spectrum of $A(t) + iB(t)$ for the inner planets (Fig. 2) where only the main peaks can be identified as combinations of the planetary secular fundamental frequencies. A quasiperiodic approximation of $A(t) + iB(t)$ of the form

$$A(t) + iB(t) \approx \sum_{k=1}^{N} \alpha_k \, e^{i(\nu_k t + \phi_k)}$$

can still be used when looking at the outer planets or, more important, for a qualitative understanding of the behaviour of the solution. With this approximation, the hamiltonian reads

$$H = \frac{\alpha}{2}(1-e(t)^2)^{-3/2}X^2 + \sqrt{1-X^2} \times \sum_{k=1}^{N} \alpha_k \sin(\nu_k t + \psi + \phi_k) \quad (5)$$

which is the hamiltonian of an oscillator of frequency αX, perturbed by a quasiperiodic external oscillation with several frequencies ν_k. Resonance will occur when $\dot{\psi} \approx \alpha X = \alpha\cos\varepsilon$ is equal to the opposite $(-\nu_k)$ of one of the frequencies ν_k. When limited to a single periodic term, and with $e(t) = Cte$, this hamiltonian is integrable and is often referred to as Colombo's top[28]. Its equilibrium points are then called Cassini states[29]. But in the present case, $e(t)$ and $A(t) + iB(t)$ take into account the secular perturbations of the whole Solar System, modelled as a system with 15 degrees of freedom, and this hamiltonian has 1 + 15 degrees of freedom. Its solutions evolve in a phase space of dimension 2 + 30 (2 accounts for the ψ, X variables). In fact, the orbital solution La90 is not coupled with the precession variables, and will thus never change. The hamiltonian is thus considered to depend on time, but in a nonperiodic way. Traditional tools, such as the Poincaré surface of sections, cannot be used for the analysis of its dynamics, and we will rely on Laskar's method of frequency analysis[8,30,31] (Box 2).

of the chaotic region changes (Fig. 5). If the obliquity of Venus reaches ~90°, it could become stabilized around that value and evade the chaotic zone. Dissipative effects could then, as the planet rotation continues to slow down, drive the spin axis towards its present value of 178°. Therefore the retrograde rotation of Venus may not be primordial. If the orientation underwent large-scale chaotic behaviour during its history, the pole of Venus could have faced the Sun for extended periods, possibly causing strong changes in the climate and atmospheric circulation.

Mars. The analysis for Mars is more direct, as its rotation speed can be considered as primordial. Mars is far from the Sun and has no large satellite whose tidal interactions could have slowed it much. Ward[15] showed that the planetary secular perturbations cause the obliquity of Mars, unlike that of the Earth, to suffer large variations of ±10° around its mean value 25°. The precession constant of Mars[16,17] (8.26″ yr^{-1}) is close to some secular frequencies of the planet, and the question of resonance has been raised several times[15,17]. Without knowledge of the precise initial conditions, no definite conclusion has been drawn.

Here we investigate the problem in a different manner, by looking at its global dynamics. First we did the frequency analysis of the obliquity of Mars over 45 Myr. We found a large chaotic zone ranging from ~0° to 60°. In Fig. 4*a* and *b*, the motion looks regular for small values of the initial obliquity. But the motion of the inner planets is chaotic[8,18], and the phases of the planetary secular terms are lost after ~100 Myr. We can thus study the problem independently of the phases. The results in Fig. 4*c*, *d* were calculated with a small change of the initial

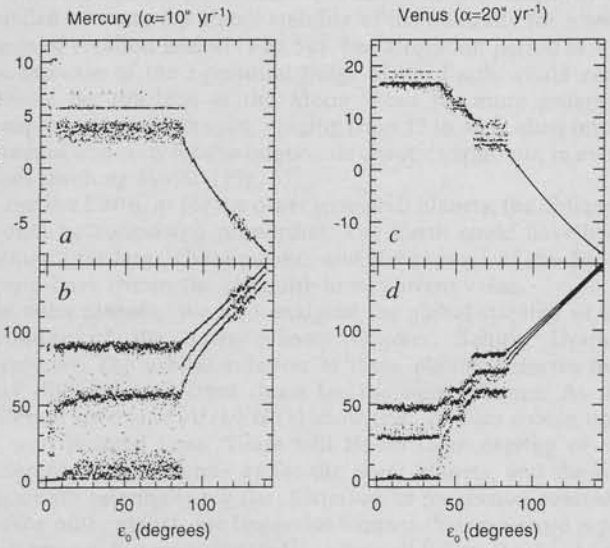

FIG. 3 Frequency analysis of the obliquity over 18 Myr for Mercury (*a, b*) with $\alpha = 10''\,\mathrm{yr}^{-1}$ (rotation period ~255 h) and Venus (*c, d*) with $\alpha = 20''\,\mathrm{yr}^{-1}$ (rotation period ~20 h). The precession frequency is plotted against the initial obliquity. The regularity of the frequency curve shows the regularity of the motion. For Mercury, the chaotic zone extends from 0° to 100°, and from 0° to nearly 90° for Venus. The maximum, mean and minimum values of the obliquity reached over 18 Myr are shown in *b* and *d*.

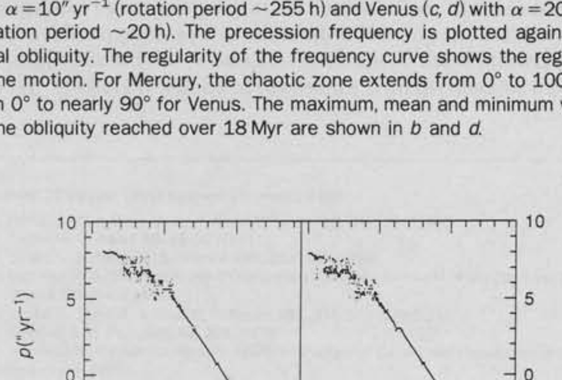

FIG. 4 Frequency analysis of the obliquity of Mars over 45 Myr. In *a* and *b*, the planetary solution is used with the present initial conditions. A large chaotic zone is visible, ranging from 0° to 60°. When the phase of the planetary solution is shifted by about −3 Myr, this large chaotic zone is even more visible (*c, d*). The maximum, mean and minimum values of the obliquity reached over 45 Myr are shown in *b* and *d*.

BOX 2 Frequency analysis

The numerical analysis of the fundamental frequencies follows the method of ref. 8. In that case, frequency analysis showed that for the inner planets (Mercury to Mars) the chaotic zone is relatively large, whereas for the outer planets (Jupiter to Neptune) this zone is much smaller. More generally, the method outlined here can be applied to study the stability of the solutions of a conservative dynamical system, and is based on a refined numerical search for a quasiperiodic approximation of its solutions over a finite time span[8,30,31]. If $f(t)$ is a function with values in the complex domain obtained numerically over a finite time span $[-T, T]$, the frequency analysis algorithm will consist of a search for a quasiperiodic approximation for $f(t)$ with a finite number of periodic terms of the form

$$\tilde{f}(t) = \sum_{k=1}^{N} a_k\, e^{i\sigma_k t}$$

The frequencies σ_k and complex amplitudes a_k are found by iteration. To determine the first frequency σ_1, one searches for the maximum anplitude of $\phi(\sigma) = \langle f(t), e^{i\sigma t}\rangle$, where the scalar product of two functions $\langle f(t), g(t)\rangle$ is defined by

$$\langle f(t), g(t)\rangle = \frac{1}{2T}\int_{-T}^{T} f(t)\bar{g}(t)\chi(t)\,dt$$

and where $\chi(t)$ is a weighting function, that is, a positive function with $1/2T\int_{-T}^{T}\chi(t)\,dt = 1$, and $\bar{g}$ the complex conjugate of g. Once the first periodic term $e^{i\sigma_1 t}$ is found, its complex amplitude a_1 is obtained by orthogonal projection, and the process is started again on the remaining part of the function $f_1(t) = f(t) - a_1 e^{i\sigma_1 t}$. A second analysis is made to estimate the precision of the determination. In the case of an integrable hamiltonian system with n degrees of freedom, the frequency analysis of the solutions will give their quasiperiodic expansion and in particular will determine the vector $(\nu_i)_{i=1,n}$ of the fundamental frequencies of the system.

If an orbit is not regular (not quasiperiodic), in case of nearly integrable systems, the frequency analysis gives a quasiperiodic approximation to the solution which holds only locally in time. In other words, it will give us a frequency vector $(\nu_i(t))_i$ for each value of t, obtained by applying the frequency analysis algorithm over the time span $[t, t+T]$. For the analysis of the precession and obliquity, we shall only look for the precession frequency p, which is associated to te angle and action variables ψ, X. The initial phase ψ_0 is fixed. If we take some initial conditions X, we can carry out the frequency analysis for the orbits corresponding to initial conditions (ψ_0, X) (at $t=0$) over the time span $[0, T]$. We thus define the frequency map

$$F_T:\ \mathbf{R} \rightarrow \mathbf{R}$$

$$X \rightarrow p$$

The regularity of the precession and obliquity can be analysed very precisely by studying the regularity of the frequency map of this problem with $1+n$ degrees of freedom. The distortions of the frequency map indicate the non-existence of invariant surfaces which may bound the obliquity variations[30,31]. As these distortions increase, they produce a complete loss of regularity of the frequency map, which is an indication of chaotic motion[31].

date of the orbital solution (~−3 Myr relative to Fig. 4*a*, *b*). This should convince the reader that the chaotic zone extends from nearly 0° to 60°. Figure 4*d* shows that some orbits changed dramatically in less than 45 Myr.

From this, we can conclude that the obliquity of the planet Mars is chaotic, with possible variations ranging from nearly 0° to ~60°. We also computed the Liapounov exponent of the obliquity of Mars for the present initial conditions[19] and found a value similar to that of the planet[18], ~1/5 Myr. We checked that this value reveals the intrinsic chaotic behaviour of the obliquity of Mars by repeating the computation with a 'regular' value (50″ yr^{-1}) of precession constant. Nevertheless, we consider the frequency analysis as more significant, as it gives the extent of the chaotic zone. Moreover, by giving the global picture of the dynamics, it shows that small changes in the initial conditions or model will not affect this result substantially.

The chaotic behaviour of the orientations of Mars could have been important in its evolution, as its obliquity could have gone

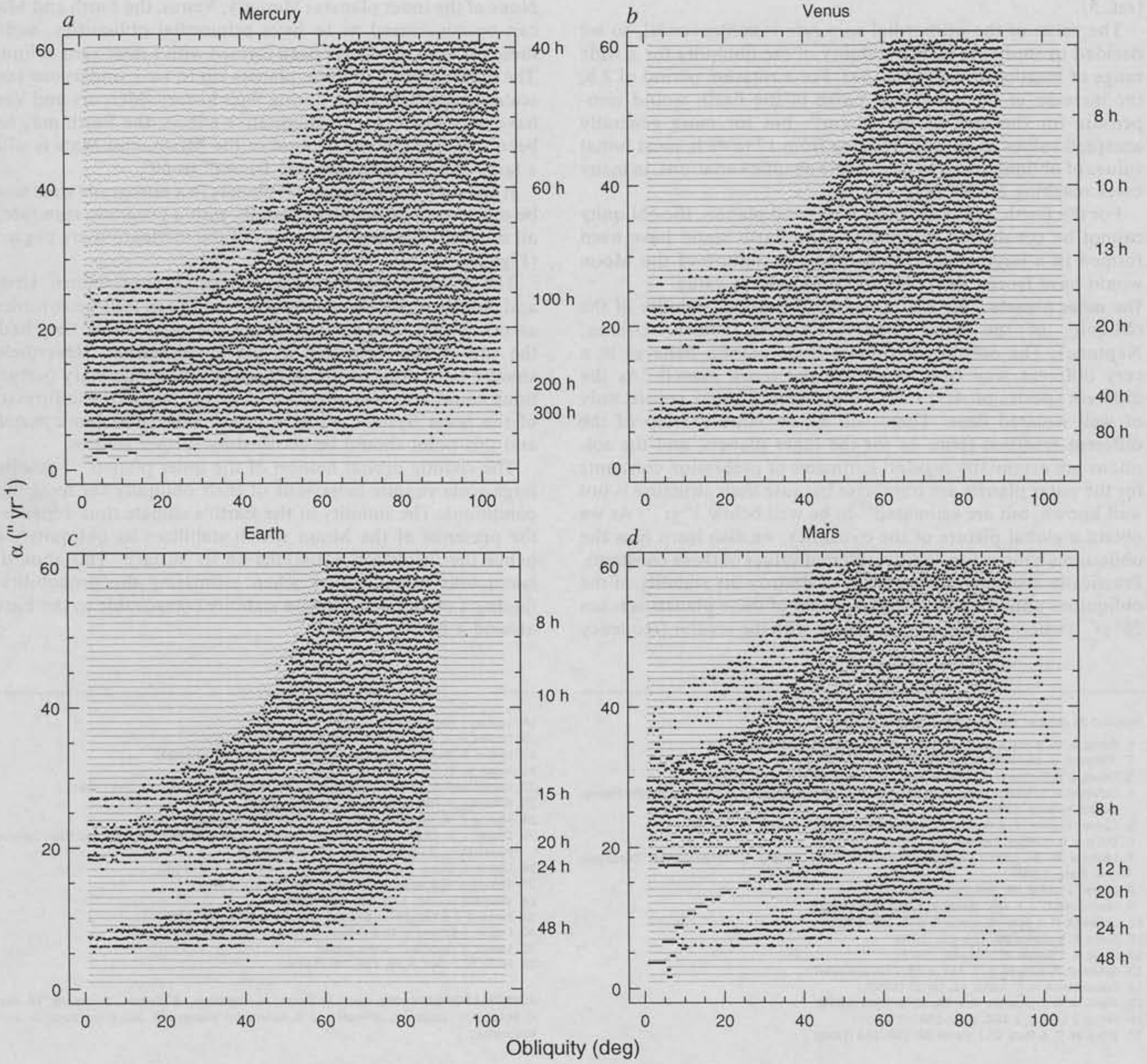

FIG. 5 The zone of large-scale chaotic behaviour for the obliquity of (*a*) Mercury, (*b*) Venus, (*c*) the Earth (without the Moon), (*d*) Mars, for a wide range of spin rate. The precession constant α is given on the left in arcseconds per year, and the estimate of the corresponding rotation period of the planet on the right, in hours. The regular solutions are represented by small dots, while large black dots represent solutions with large-scale chaotic behaviour. The chaotic motion is estimated by the diffusion of the precession frequency obtained from numerical frequency analysis over 36 Myr. For each initial condition (ε, α), the corresponding point is shifted vertically proportionally to the diffusion of the orbit. In the large chaotic zones visible here, the chaotic diffusion will occur on horizontal lines (α is fixed), and the obliquity of the planet can explore horizontally all the black zone in a few million years. The extent of the chaotic zone should be even larger when one considers diffusion over longer timescales. With the Moon, one can consider that the present situation of the Earth can be represented in *c* roughly by the point $\varepsilon = 23°$, $\alpha = 55″ \text{yr}^{-1}$, which is in the middle of a large zone of regular motion. Without the Moon, for a spin period ranging from about 12 h to 48 h, the obliquity of the Earth would suffer very large chaotic variations.

up to 60° during its history. The obliquity of Mars cannot be considered as primordial; the planet also could have been formed with any obliquity from 0° to 60°, and been driven to its present values only by the effect of the secular planetary perturbations.

The Earth. The equations for the precession of the Earth are complicated by the presence of the Moon; we consider the stability of its obliquity in the accompanying paper[5]. The present obliquity variations[20] are only ±1.3° around the mean value of 23.3°, but in the absence of the Moon, the torque exerted on the Earth would be smaller; we found, assuming a primordial rotation of 15 h, that the obliquity of the Earth would show very large variations in a chaotic zone ranging from nearly 0° to 85° (ref. 5).

The value of the primordial spin rate is controversial, so we decided to study the global stability of the obliquity for a wide range of rotation period (Fig. 5*c*). For a rotation period of 8 h, the increase of the equatorial bulge of the Earth would compensate for the lack of the Moon[21], but for more generally accepted values of its spin, ranging from 12 to 48 h, most initial values of obliquity lead to large-scale chaotic variations, in many cases reaching 80–90° (Fig. 5).

For the Earth, as for the other terrestrial planets, the obliquity cannot be considered primordial. The Earth could have been formed in a large chaotic zone, and the capture of the Moon would have frozen the obliquity to its current value.

The outer planets. We also analysed the global stability of the obliquity of the outer planets (Jupiter, Saturn, Uranus, Neptune). The orbital solution of these planets behaves in a very different way from those for the inner planets. As the different spectra of $A(t)+iB(t)$ show (Fig. 2), they consist only of well isolated lines. There will be no large overlap of the different resonant terms as for the inner planets, and the solutions are essentially regular. Estimates of precession constants for the outer planets are imprecise because their structure is not well known, but are estimated[22] to be well below $5'' \mathrm{yr}^{-1}$. As we obtain a global picture of the dynamics, we also learn how the obliquities behave for slightly different values of these constants. Practically speaking, it is difficult to destroy the stability of the obliquities until the precession constant of these planets reaches $26'' \mathrm{yr}^{-1}$, where it will be in resonance with the secular frequency s_6 (ref. 8). Even then, this resonance being isolated, the motion will show large but regular oscillations. We can thus say that, as for the orbital motion, the obliquities of the outer planets are essentially stable, and should be considered as primordial.

But primordial here means that the obliquities have not changed since the Solar System was in the final state of its formation, in a state similar to its present one (at least for the outer planets). If it had, at some previous stage, been more massive than now, similar instabilities as for the inner planets could have occurred, which may have driven the obliquities of the outer planets to their present configurations.

Conclusions

None of the inner planets (Mercury, Venus, the Earth and Mars) can be considered as to have primordial obliquities, and all these planets could have been formed with a near-zero obliquity. The obliquities of all these planets could have undergone large-scale chaotic behaviour during their history. Mercury and Venus have been stabilized by dissipative effects, the Earth may have been stabilized by the capture of the Moon, and Mars is still in a large chaotic zone, ranging from 0° to 60°.

It should also be noted that planets in a retrograde state would be much more stable than planets with a prograde spin rate, as all the main forcing frequencies of the inclination are negatives (Fig. 2).

The obliquities of the outer planets (Jupiter, Saturn, Uranus and Neptune) are essentially stable, and can thus be considered as primordial, that is, with about the same value they had at the end of the formation of the Solar System. Nevertheless, chaotic behaviour of the obliquities under planetary perturbations could have occurred in an earlier stage of the formation of the Solar System, when it could have been more massive, and this point should be clarified by further studies.

The chaotic orbital motion of the inner planets[8,18] results in large-scale chaotic behaviour of their obliquity for most initial conditions. The stability of the Earth's climate thus depends on the presence of the Moon which stabilizes its obliquity[5], and hence the insolation variations on its surface. This should be taken into consideration when estimating the probability of finding a planet with climate stability comparable to the Earth's around a nearby star. □

Received 26 October 1992; accepted 25 January 1993.

1. Harris, A. W. & Ward, W. R. *A. Rev. planet. Sci.* **10,** 61–108 (1982).
2. Tremaine, S. *Icarus* **89,** 85–92 (1991).
3. Dones, L. & Tremaine, S. *Science* **259,** 350–354 (1993).
4. Safronov, V. S. *Evolution of the Protoplanetary Cloud and Formation of the Earth and the Planets* (Nauka, Moscow, 1969).
5. Laskar, J., Joutel, F. & Robutel, P. *Nature* **361,** 615–617 (1993).
6. Chirikov, B. V. *Phys. Rep.* **52,** 263 (1979).
7. Lambeck, K. *The Earth's Variable Rotation: Geophysical Causes and Consequences* (Cambridge Univ. Press, 1980).
8. Laskar, J. *Icarus* **88,** 266–291 (1990).
9. MacDonald, G. J. F. *Rev. Geophys.* **2,** 467–541 (1964).
10. Goldreich, P. & Soter, S. *Icarus* **5,** 375–389 (1966).
11. Burns, J. A. *Icarus* **28,** 453–458 (1976).
12. Peale, S. J. *Icarus* **28,** 459–467 (1976).
13. Goldreich, P. & Peale, S. J. *Astr. J.* **75,** 273–284 (1970).
14. Dobrovolskis, A. R. *Icarus* **41,** 18–35 (1980).
15. Ward, W. R. *J. geophys. Res.* **79,** 3375–3386 (1974).
16. Hilton, J. L. *Astr. J.* **102,** 1510–1527 (1991).
17. Ward, W. R. & Rudy, D. J. *Icarus* **94,** 160–164 (1991).
18. Laskar, J. *Nature* **338,** 237–238 (1989).
19. Davies, M. E. *et al. Celest. Mech.* **53,** 377–379 (1992).
20. Laskar, J., Joutel, F. & Boudin, F. *Astr. Astrophys.* (in the press).
21. Ward, W. R. *Icarus* **50,** 444–448 (1982).
22. Korycansky, D. G., Bodenheimer, P. & Pollack, J. B. *Icarus* **92,** 234–251 (1991).
23. Goldreich, P. *Rev. Geophys.* **4,** 411–439 (1966).
24. Murray, C. A. *Vectorial Astrometry* (Hilger, Bristol, 1983).
25. Laskar, J. in *Chaos, Resonance and Collective Dynamical Phenomena in the Solar System* (ed. Ferraz-Mello, S.) 1–16 (Kluwer, Dordrecht, 1992).
26. Laskar, J., Quinn, T. & Tremaine, S. *Icarus* **95,** 148–152 (1992).
27. Sussman, G. & Wisdom, J. *Science* **257,** 56–62 (1992).
28. Colombo, G. *Astr. J.* **71,** 891–896 (1966).
29. Henrard, J. & Murigande, C. *Celest. Mech.* **40,** 345–366 (1987).
30. Laskar, J. Froeschlé, C. & Celletti, A. *Physica D***56,** 253–269 (1992).
31. Laskar, J. *Physica D* (in the press).
32. Peale, S. J. *Astr. J.* **79,** 722–744 (1974).

ACKNOWLEDGEMENTS. We thank P. Barge, A. Chenciner, S. Dumas, D. Gautier, M. Herman, F. Joutel, P. Y. Longaretti, S. Peale and B. Sicardy for discussions, and J. Marchand for technical assistance.