VSS from Distributed ZK Proofs and Applications

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Abstract. Non-Interactive Verifiable Secret Sharing (NI-VSS) is a technique for distributing a secret among a group of individuals in a verifiable manner, such that shareholders can verify the validity of their received share and only a specific number of them can access the secret. VSS is a fundamental tool in cryptography and distributed computing. In this paper, we present an extremely efficient NI-VSS scheme using Zero-Knowledge (ZK) proofs on secret shared data. While prior VSS schemes have implicitly used ZK proofs on secret shared data, we specifically use their formal definition recently provided by Boneh et al. in CRYPTO 2019. The proposed NI-VSS scheme uses a quantum random oracle and a quantum computationally hiding commitment scheme in a black-box manner, which ensures its ease of use, especially in post-quantum threshold protocols. Implementation results further solidify its practicality and superiority over current constructions. With the new VSS scheme, for parameter sets $(n, t) = (128, 63)$ and $(2048, 1023)$, a dealer can share a secret in less than 0.02 and 2.0 *seconds*, respectively, and shareholders can verify their shares in less than 0.4 and 5.0 *milliseconds*. Compared to the well-established Pedersen VSS scheme, for the same parameter sets, at the cost of slightly higher communication, the new scheme is respectively 22.5× and 3.25× faster in the sharing phase, and notably needs 271× and 479× less time in the verification. Leveraging the new NI-VSS scheme, we revisit several classic and PQ-secure threshold protocols and improve their efficiency. Our revisions led to more efficient versions of both the Pedersen DKG protocol and the GJKR threshold signature scheme. We show similar efficiency enhancements and improved resilience to malicious parties in isogeny-based DKG and threshold signature schemes. We think, due to its remarkable efficiency and ease of use, the new NI-VSS scheme can be a valuable tool for a wide range of threshold protocols.

Keywords: Verifiable Secret Sharing · ZK Proofs on Secret Shared Data · Shamir Secret Sharing · DKG · Threshold Signatures · Isogenies.

1 Introduction

Secret sharing schemes are fundamental tools in the field of threshold cryptography and secure multi-party computation. Such schemes consist of a sharing

phase, where a dealer shares a secret among the shareholders, followed by a reconstruction phase where qualified shareholders collaborate to reconstruct the original secret. Standard secret sharing schemes, such as Shmair's protocol [32], assume the presence of honest parties but do not provide security against malicious participants. Verifiable Secret Sharing (VSS) schemes [14, 19] have been developed to address the challenges posed by malicious players. These schemes aim to withstand various attacks, including incorrect share distribution by the dealer and malicious behavior by the shareholders (e.g., using incorrect shares) during the reconstruction phase. Depending on the communication model, to incorporate verifiability, typically interaction among the dealer and shareholders is required. It can however be shown that, assuming the dealer has a broadcast channel, a single message from the dealer to the shareholders can be sufficient. This is known as a Non-Interactive VSS (NI-VSS).

Most existing constructions of VSS schemes are based on regular secret-sharing schemes, often starting with Shamir's scheme [32], and then adding verifiability features on top [4, 14, 19, 22–24, 28, 31]. The known discrete-logarithm (DL) based VSS schemes such as those by Feldman [19], Pedersen [28], Schoenmakers [31], and their variants, utilize Shamir secret sharing and achieve verifiability by having the dealer publish the shares and coefficients of the underlying secret polynomial in the group. Then, they leverage the homomorphic property of the group to convince the shareholders [19, 28], or an external verifier [31], that the secret sharing is performed correctly. DL-based VSS schemes are typically non-interactive and support public verifiability, allowing both shareholders and external verifiers to verify the validity of the shares without interaction. However, due to the threat posed by Shor's algorithm [33], discrete logarithm based VSS schemes are not suitable for cryptographic protocols (e.g., distributed key generation schemes, threshold signatures, etc.), that require post-quantum security. Gentry, Halevi, and Lyubashevsky [23] recently proposed a practical non-interactive publicly VSS scheme that relies on lattice-based and DL-based problems at the same time, unfortunately making it unsuitable for use in post-quantum secure threshold protocols. Given the limitations and vulnerabilities of existing NI-VSS schemes, it becomes imperative to develop an efficient post-quantum secure VSS scheme that can also address challenges of scalability and computational overhead. Such VSS schemes can pave the way for the realization of more efficient post-quantum threshold protocols.

In the VSS scheme proposed by Ben-or, Goldwasser, and Wigderson (BGW) [4], a dealer employs a distributed Zero-Knowledge (ZK) proof scheme based on bivariate polynomials to add (designated) verifiability to the Shamir secret sharing scheme. The BGW VSS scheme achieves Information-Theoretical (IT) security and can be employed in both classical and post-quantum secure threshold protocols. However, in their (non-interactive) ZK proof scheme the verifiers need to interact two-by-two for share validation and to achieve (perfect) soundness their scheme requires at least two-thirds of the designated verifiers to be honest.

In Crypto 2019, Boneh et al. [10] provided a formal definition for ZK proofs over secret shared data and presented several feasibility and infeasibility results.

In a ZK proof scheme over secret shared data, there is a single prover P and n (designated) verifiers $\{V_i\}_{i=1}^n$, and each verifier V_i holds a piece (share) x_i of an input (statement) x, which is distributed among n participants. The prover's task is to convince the (designated) verifiers that the main input x belongs to a specific language L. Essentially, the prover P possesses full knowledge of x, while each verifier V_i possesses a secret share denoted as x_i. In their best feasibility (and positive) result, Boneh et al. [10] demonstrated that, in the majority honest setting, using a robust encoding scheme, any multi-round public-coin linear Interactive Oracle Proof (IOP) for a non-distributed relation R_L can be compiled into a secure ZK proof scheme over secret shared data. The resulting distributed ZK proof scheme satisfies (computational) soundness against the prover and $t < n/2$ malicious verifiers. Moreover, it guarantees ZK even if t of the verifiers collude [10, Section 6.3], where t represents a threshold parameter in the underlying encoding scheme. Boneh et al. [10] coined the term "Strong zero-knowledge" to describe this variant of ZK, which ensures that even if up to t verifiers collude, they learn nothing about the witness. We will refer to this notion as "threshold zero-knowledge" (TZK) in this paper.[3] Based on the formal definitions, we can restate that the BGW VSS scheme [4] has been proven to achieve TZK and (perfect) soundness against the prover, given that at least two-thirds of the (designated) verifiers are honest.

Consequently, a generic approach for constructing a ZK proof scheme over secret shared data for n-distributed relations R_i involves first developing a multi-round public-coin IOP for the non-distributed relation R. Subsequently, Boneh et al.'s compiler can be utilized to transform it into a distributed ZK proof scheme, featuring a single prover P and n designated verifiers $\{V_i\}_{i=1}^n$. However, it is worth noting that generic approaches are typically less efficient compared to ad-hoc constructions tailored for specific purposes in practical implementations.

Our Contributions. We summarize the contributions of this paper as follows:

An Efficient Post-Quantum Secure NI-TZK for the Shamir Relation. Considering the feasibility result of Boneh et al. [10], we directly (without using their compiler [10]) construct an efficient Non-Interactive TZK (NI-TZK) proof scheme for the n-distributed relations $R_1, \ldots, R_n$, where

$$R_i = \{(x_i, f(X)) | f(i) = x_i\}. \tag{1}$$

Here $f(X) \in \mathbb{Z}_N[X]_t$ is a secret polynomial in X of degree (at most) t and with coefficients defined over the ring $\mathbb{Z}_N$. The proposed construction is built in the majority honest setting (i.e., the majority of the verifiers are honest) and utilizes a quantum computationally hiding commitment scheme. We prove (in Theorem 1) that in the Quantum Random Oracle Model (QROM), the proposed

[3] This choice is done for two main reasons. First, *strong* ZK might lead to misunderstandings, as the majority of verifiers is actually honest in our cases. Second, we risk confusion by using the abbreviation SZK, which usually stands for *statistical* zero-knowledge.

NI-TZK proof scheme satisfies completeness, TZK, and soundness against the prover and t malicious verifiers, as formally defined in [10].

NI-VSS Schemes from NI-TZK Proofs and a Quantum Secure Scheme. We further show how one can use a secure NI-TZK proof scheme for the n-distributed relations given in equation (1), and build a *computationally secure* NI-VSS scheme based on Shamir secret sharing in the majority honest setting. Building upon that, we use the proposed NI-TZK proof scheme and present an extremely efficient computationally secure NI-VSS scheme that works over general rings, and is proven to be secure in the QROM. One notable factor contributing to the efficiency of new VSS scheme is using lightweight cryptographic operations such as hashing and polynomial evaluation, in the underlying NI-TZK proof scheme. Within the new VSS scheme, we introduce a novel reconstruction approach that, in scenarios where the dealer is one of the parties (e.g., as in the DKG protocols and threshold signatures), can lead to the development of more efficient threshold protocols.

Our resulting NI-VSS scheme serves as a post-quantum secure alternative to the classical Pedersen VSS scheme [28] (or Feldman's VSS scheme [19]) when public verifiability is unnecessary. We later show that this scenario often occurs in various threshold protocols. It can also be considered an alternative to the Information-Theoretically (IT) secure BGW VSS [4] in cases where quantum computational security suffices and there is a desire to reduce communication between parties, or when assuming two-thirds of the parties are honest among the shareholders is challenging. Table 1 provides a comprehensive summary of the key features of our proposed NI-VSS scheme, comparing it to the well-known Pedersen [28] and BGW [4] VSS schemes from various perspectives.

To assess the empirical performance of new VSS scheme, we implemented a prototype of it alongside the Pedersen scheme [28] using SageMath. Implementation results show that using the new NI-VSS scheme a dealer can share a secret with 2048 parties in 2 seconds, and shareholders can verify the validity of their shares in less than 5 milliseconds. When considering the same number of parties and aiming for 128-bit quantum security, the dealer broadcasts a

Table 1. A comparison among BGW [4], Pedersen [28] and the new NI-VSS schemes.

BGW VSS [4]	**Pedersen VSS [28]**	**This Work**
uses bivariate polynomials (lightweight operations)	based on discrete logarithm (heavy operations)	based on hash functions (lightweight operations)
achieves Information Theoretical (IT) security	achieves IT and classic (computational) security	achieves post-quantum (computational) security
needs $\geq \frac{2}{3}$ honest parties	needs $\geq \frac{1}{2}$ honest parties	needs $\geq \frac{1}{2}$ honest parties
verification is designated and to verify the shares the verifiers need to interact two-by-two, which induces $O(n)$ communication for each shareholder	verification is designated (but can also be made public) and to verify the shares the verifiers do not need to interact two-by-two	verification is designated and to verify the shares the verifiers do not need to interact two-by-two

proof of approximately 130KB and privately sends less than 32B to each shareholder. Our empirical analysis affirm the superiority of our NI-VSS scheme over the well-known Pedersen scheme in both sharing and verification steps, in addition to post-quantum security. The new NI-VSS scheme demonstrates significant efficiency improvements in the verification phase, achieving speedups of approximately 271×, 437×, and 498× compared to the Pedersen scheme for (n, t) values of $(128, 63)$, $(512, 255)$, and $(8194, 4095)$, respectively. In the sharing phase under the same settings, our scheme is about 22.6×, 9.3×, and 1.57× faster than the Pedersen scheme. In terms of communication costs, when compared to the Pedersen scheme, our VSS scheme results in a slight increase in broadcast communication for the dealer. On the positive side, it reduces the dealer's private communication.

We believe, the simplicity and efficiency of our new NI-VSS scheme can make it an attractive choice for various large-scale threshold protocols, especially those that require post-quantum security.

A Concurrent Work in the Asynchronous Setting. In concurrent work, Shoup and Smart [34] also introduced a novel VSS scheme that employs lightweight cryptographic operations, such as polynomial evaluation and hashing (for a random oracle/beacon). Shoup and Smart's scheme supports batched secret sharing, is tailored for the asynchronous setting, and requires the participation of at least 2/3 of the parties to be honest. Batched secret sharing, allows the dealer to save on computational and communication costs in case of sharing multiple secrets with the same parties. In this study, our focus is on the case where the dealer shares a single secret with the parties. When comparing our scheme to BGW [4] and the Shoup-Smart scheme [34], there are certain trade-offs that can be summarized as follows:

- **Ben-Or, Goldwasser, and Wigderson (BGW)** [4]:
  (i) Utilizes lightweight operations, specifically polynomial evaluations.
  (ii) Designed for the synchronous communication model.
  (iii) Achieves Information-Theoretic (IT) Security.
  (iv) Requires at least 2/3 honest parties.
  (v) To verify the shares, verifiers *need* to interact pairwise, which induces $O(n)$ communication for each shareholder.
- **This Work** (Section 3.2):
  (i) Utilizes lightweight operations, i.e., polynomial evaluations and hashing.
  (ii) Designed for the synchronous communication model.
  (iii) Achieves Post-Quantum (PQ) Computational Security.
  (iv) Requires 1/2 honest parties
  (v) Verifiers *do not need* to interact pairwise to verify the shares, leading to shorter communication compared to BGW.
- **Shoup-Smart** [34]:
  (i) Utilizes lightweight operations, i.e., polynomial evaluations and hashing.
  (ii) Designed for the asynchronous communication model.
  (iii) Achieves Post-Quantum (PQ) Computational Security.
  (iv) Requires at least 2/3 honest parties.
  (v) Verifiers *do not need* to interact pairwise to verify the shares, resulting in shorter communication compared to BGW.

Application Examples of New NI-VSS Scheme. As our next major contribution, we leverage the new NI-VSS scheme and revisit several threshold protocols based on discrete logarithm and isogenies, and improve their efficiency, and in some cases also decrease the lower bound on the number of honest parties.

More Efficient Threshold Protocols in the DL Setting. As mentioned previously, the new VSS scheme serves as a more efficient alternative to the well-established Pedersen VSS scheme [28] when public verifiability is not required. Our observations indicate that this scenario commonly arises in Distributed Key Generation (DKG) protocols and threshold signatures. In light of this insight, we revisit Pedersen's DKG protocol [27] in conjunction with the robust threshold signature scheme proposed by Gennaro, Jarecki, Krawczyk, and Rabin (GJKR) [21], which employs Schnorr's signature for signing and Pedersen's DKG protocol for the generation of (secret, public, and ephemeral) keys. To this end, we first build an NI-TZK proof scheme for the following set of n-distributed relations,

$$R_i = (y, x_i, f(X))|y = g^{f(0)} \wedge f(i) = x_i, \quad i = 1, \ldots, n \tag{2}$$

where $f(X) \in \mathbb{Z}_q[X]_t$ is a secret polynomial in X of degree (at most) t with coefficients defined over the field $\mathbb{Z}_q$. This NI-TZK proof scheme serves as the main component for our revisions and holds potential interest in other DL-based threshold protocols that utilize Shamir secret sharing. Subsequently, we present a new DKG protocol and a Schnorr-based threshold signature scheme, which can be considered as more efficient alternatives to Pedersen's DKG protocol [27] and the GJKR threshold signature [21].

When comparing our resulting variants to the original schemes, there are certain trade-offs. While our variants slightly increase communication costs, they improve computational efficiency in both schemes. In summary, in our proposed DKG protocol, each party needs to perform approximately $2n$ exponentiations in the group, $5n$ (short) hashes, and $3n$ degree-t polynomial evaluations in the field. This represents a factor of $t \approx n/2$ improvement compared to the secure version of Pedersen DKG protocol [21], which demands roughly $2tn+2n$ exponentiations in the group and $2n$ degree-t polynomial evaluations in the field. While incurring slightly higher communication costs, our new threshold signature scheme offers similar improvements over the GJKR threshold signature scheme [21]. The detailed comparisons are provided in Sec. 4 (and Table 4).

More Efficient Threshold Protocols from Isogenies. As previously discussed, the new NI-VSS scheme can also be integrated into various post-quantum secure threshold protocols. Notably, it can serve as an alternative to the BGW VSS [4] in certain scenarios. By adopting the new VSS scheme in place of BGW, it becomes possible to reduce communication costs and improve tolerance for malicious parties, albeit at the expense of transitioning from information-theoretic (IT) security to quantum computational security. Taking this into consideration, we revisit the isogeny-based DKG protocol developed by Atapoor, Baghery, Cozzo, and Pedersen [2], alongside the CSI-FiSh-based threshold signature scheme introduced by Campos and Muth [12].

Currently, their DKG protocol [2] stands as the most efficient scheme in terms of isogeny computations within the CSIDH (Commutative Supersingular Isogeny Diffie-Hellman) setting [13]. We show that by integrating the new NI-VSS scheme into the VSS step of their DKG protocol, we can address two bottlenecks present in their scheme. Specifically, we reduce the requirement from needing at least 2/3 honest shareholders to a more practical threshold of just 1/2. Additionally, we eliminate the need for pairwise interactive verification, which was a primary reason to the high communication overhead in the VSS step of their DKG protocol. While these enhancements do come at the cost of sacrificing IT security in the VSS step in favor of quantum computational security, it's worth noting that the DKG protocols proposed in [2] rely on quantum computational security from the outset. These improvements can make the revisited DKG protocol highly appealing for use in CSIDH-based threshold settings. The resulting DKG protocol retains the same efficiency in terms of isogeny computations.

The threshold signature scheme proposed by Campos and Muth [12] is based on the basic version of the CSI-FiSh signature [8], which features shorter public keys but longer signature sizes, as well as slower signing and verification algorithms. To enhance the efficiency of their threshold signature, we introduce two key modifications. First, we adapt their scheme to work with the CSI-SharK signature [1], which has been demonstrated to outperform CSI-FiSh in the threshold setting. This modification enables us to leverage our revisited DKG protocol, resulting in more efficient key generation for the resulting robust threshold signature scheme. Furthermore, we apply a similar strategy used in the construction of the revisited DKG protocol to improve the efficiency of ephemeral key generation in the resulting threshold signature. This further enhances the efficiency of the distributed signing protocol, leading to the development of a new and more efficient isogeny-based threshold robust signature scheme.

Outline. In Sec. 2, we provide an overview of some preliminary concepts. In Sec. 3, we first present our NI-VSS scheme, and then evaluate its performance through a prototype implementation. Leveraging the proposed NI-VSS scheme, in Sec. 4, we revisit the well-known Pedersen DKG protocol [28] and the GJKR threshold signature scheme [21], introducing new variants that offer improved efficiency. Similarly, in Sec. 5, we revisit two isogeny-based DKG protocols [2] and a threshold signature scheme [12], and present two new versions that offer improved efficiency. Finally, in Sec. 6, we conclude the paper.

2 Preliminaries

Notation. We let λ denote a security parameter. A function is called *negligible in* X, written $\mathsf{negl}(X)$, if for any constant c, there exists some X_0, such that $f(X) < X^{-c}$ for $X > X_0$. A function that is negligible in the security parameter λ is simply called negligible. We use the assignment operator $\leftarrow$ to denote uniform sampling from a set Ξ, e.g. $x \leftarrow \Xi$. We write $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}_N[X]_t$ for polynomials of degree t in the variable X and with coefficients in $\mathbb{Z}_N$. For

$n \in \mathbb{N}$, we write $[n] = \{1, \ldots, n\}$. Finally all logarithms are in base 2.

We also introduce the notion of *exceptional sets*, which occur naturally when working over rings $\mathbb{Z}_N$.

Definition 1 (Exceptional set [3, 9, 15]). *An* exceptional set (modulo N) *is a set $\Xi_k = \{c_1, \ldots, c_k\} \subseteq \mathbb{Z}_N$, where the pairwise difference of all distinct elements is invertible modulo N. If further the pairwise sum of* all *elements is invertible modulo N, Ξ_k is called a* superexceptional set (modulo N).

2.1 Zero-Knowledge Proofs on Secret Shared Data

In typical NIZK (non-interactive zero-knowledge) arguments for NP languages, there is a single prover P and a single verifier V, where P knows both a statement x and witness for the statement w, while V only knows the statement x. In CRYPTO 2019, Boneh et al. [10], presented formal definitions for distributed ZK proofs which a prover interacts with several verifiers $\{V_i\}_{i=1}^n$ over a network that includes secure point-to-point channels. In such a model, each verifier V_j holds a piece (share) $x^{(j)} \in \mathbb{F}^{l_j}$ of an input (statement) x, and the prover's task is to convince the verifiers that the main input x is in some language $L \subseteq \mathbb{F}^l$.

Similar to the typical cases, such proof systems must be complete, meaning that if $x \in L$, an honest prover will be able to convince honest verifiers. Similarly, they should satisfy soundness, meaning that if $x \notin L$, then all verifiers will reject the verification except for a negligible probability. However, in certain settings, including ours, a limited number of verifiers may be malicious and collude with the adversarial prover. In such cases, the malicious verifiers might accept a fake proof. Finally, the proof system must satisfy a variant of ZK, so called *threshold ZK*, as introduced (as strong ZK) by Boneh et al. [10]. TZK implies that any subset of the verifiers up to a certain bound should learn no additional information about statement x, beyond their own shares and the fact that $x \in L$. Note that in standard ZK, the verifier learns the statement x and the fact that $x \in L$, but in threshold ZK, a single verifier only learns his share of x and the fact that $x \in L$. In other words, a set of verifiers only learn that they are jointly holding pieces (shares) of $x \in L$.

Definition 2 (Distributed Inputs, Languages, and Relations [10]). *Let n be a number of parties, $\mathbb{F}$ be a finite field, and $l, l_1, l_2, \cdots, l_n \in \mathbb{N}$ be length parameters, where $l = l_1 + l_2 + \cdots + l_n$. An n-distributed input over $\mathbb{F}$ (or just distributed input) is a vector $x = x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)} \in \mathbb{F}^l$ where $x^{(i)} \in \mathbb{F}^{l_i}$, and it refers to a piece (or share) of x. An n-distributed language L is a set of n-distributed inputs. A distributed NP relation with witness length h is a binary relation $R(x, w)$ where x is an n-distributed input and $w \in \mathbb{F}^h$. We assume that all x in L and $(x, w) \in R$ share the same length parameters. Finally, we let $L_R = \{x : \exists w (x, w) \in R\}$.*

Next, we recall the formal definition provided by Boneh et al. [10] for ZK proofs over shared data which originally are defined over a field. In some cases,

we employ an extended version of their definitions that naturally encompasses rings. In this model, parties can have synchronous communication over secure point-to-point channels.

Definition 3 (n-Verifier Interactive Proofs [10]). *An* n-Verifier Interactive Proof *protocol over* $\mathbb{F}$ *is an interactive protocol* $\Pi = (P, V_1, V_2, \cdots, V_n)$ *involving a prover* P *and* n *verifiers* $\{V_i\}_{i=1}^n$*. The protocol proceeds as follows.*

- *In the beginning of the protocol the prover* P *holds an* n*-distributed input* $x = x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)} \in \mathbb{F}^l$*, a witness* $w \in \mathbb{F}^h$*, and each verifier* V_j *holds an input piece (or share)* $x^{(j)}$*.*
- *The protocol allows the parties to communicate in synchronous rounds over secure point-to-point channels. While honest parties send messages according to* Π*, malicious parties (i.e., adversary) can send arbitrary messages.*
- *At the end, each verifier outputs either* 1 *(accept) or* 0 *(reject) based on its view, where the view of* V_j *consists of its input piece* $x^{(j)}$*, random input* $r^{(j)}$*, and messages it received during the protocol execution.*

In the rest, $\Pi(x, w)$ denotes running Π on shared input x and witness w, and says that $\Pi(x, w)$ accepts (respectively, rejects) if at the end all verifiers output 1 (resp., 0). $View_{\Pi,T}(x, w)$ denotes the (joint distribution of) views of verifiers $\{V_j\}_{j\in T}$ in the execution of Π on the distributed input x and witness w.

Let $R(x, w)$ be a k-distributed relation over finite field $\mathbb{F}$. We say that an n-verifier interactive proof protocol $\Pi = (P, V_1, \cdots, V_n)$ is a distributed threshold ZK proof protocol for R with t-security against malicious prover *and* malicious verifiers, and with soundness error ϵ, if Π satisfies the following properties [10]:

Definition 4 (Completeness). *For every* n*-distributed input* $x = x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)} \in \mathbb{F}^l$*, and witness* $w \in \mathbb{F}^h$*, such that* $(x, w) \in R$*, the execution of* $\Pi(x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)}, w)$ *accepts with probability 1.*

Definition 5 (Soundness Against Prover and t Verifiers.). *For every* $T \subseteq [n]$ *of size* $|T| \leq t$*, an* $\mathcal{A}$ *controlling the prover* P *and verifiers* $\{V_j\}_{j\in T}$*,* n*-distributed input* $x = x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)} \in \mathbb{F}^l$*, and witness* $w \in \mathbb{F}^h$ *the following holds. If there is no* n*-distributed input* $x' \in L_R$ *such that* $x'_H = x_H$*, where* $H = [n]/T$*, the execution of* $\Pi^\star(x, w)$ *rejects except with at most* ϵ *probability, where here* $\Pi^\star$ *denotes the interaction of* $\mathcal{A}$ *with the honest verifiers.*

Definition 6 (Threshold ZK). *For every* $T \subseteq [n]$ *of size* $|T| \leq t$ *and an* $\mathcal{A}$ *controlling* $\{V_j\}_{j\in T}$*, there exists a simulator* $\mathcal{S}$ *such that for every* n*-distributed input* $x = x^{(1)} \parallel x^{(2)} \parallel \cdots \parallel x^{(n)} \in \mathbb{F}^l$*, and witness* $w \in \mathbb{F}^h$ *such that* $(x, w) \in R$*, we have* $\mathcal{S}((x^{(j)})_{j\in T}) \equiv View_{\Pi^\star,T}(x, w)$*. Here,* $\Pi^\star$ *denotes the interaction of adversary* $\mathcal{A}$ *with the honest prover* P *and the honest verifiers* $\{V_j\}_{j\in T}$*.*

Remark 1 (Threshold Honest-Verifier ZK). In the context of Threshold ZK, one may consider a relaxed definition, Threshold *Honest-Verifier* ZK, that retains the same properties as the original definition, with the added requirement that the subset of verifiers, $\{V_j\}_{j\in T}$, is stipulated to follow the protocol honestly.

2.2 Verifiable Secret Sharing

Secret sharing is a technique for securely distributing a secret among a group of parties, where no single party can learn the secret individually. However, when a sufficient number of parties come together and combine their 'shares', the original secret can be reconstructed. Throughout the paper, our studied protocols use Shamir Secret Sharing [32] for securely sharing a secret, which we review below.

Shamir Secret Sharing. A $(t+1, n)$-Shamir secret sharing scheme [32] allows n parties to individually hold a share x_i of a common secret x_0, such that any subset of t parties or less are not able to learn any information about the secret x_0, while any subset of at least $t+1$ parties are able to efficiently reconstruct the common secret x_0. In more detail, this is achieved via polynomial interpolation over the ring $\mathbb{Z}_N$. A common polynomial $f(x) \in \mathbb{Z}_N[x]_t$ is chosen, such that the secret x_0 is set to be its constant term, namely $x_0 = f(0)$. Each party P_i for $i \in \{1, \cdots, n\}$ is assigned the secret share $x_i = f(i)$. Then any subset $Q \subseteq \{1, \ldots, n\}$ of at least t parties can reconstruct the secret x_0 via Lagrange interpolation by computing $x_0 = f(0) = \sum_{i \in Q} x_i \cdot L_{0,i}^{Q}$, where

$$L_{0,i}^{Q} := \prod_{j \in Q \setminus \{i\}} \frac{j}{j-i} \pmod{N}.$$

are the Lagrange basis polynomials evaluated at 0. Any subset of less than t parties are not able to find $x_0 = f(0)$, as this is information theoretically hidden from the other shares. In the case where $\mathbb{Z}_N$ is a ring, the difference of any elements in $\{1, \ldots, n\}$ must be invertible modulo N, thus $\{1, \ldots, n\}$ must be an exceptional set. This is only the case if n is smaller than the smallest prime divisor q of N. In the case where more than q parties want to participate in the protocol, we would have to work in a subgroup $\mathbb{Z}_{N'} \subset \mathbb{Z}_N$ such that the smallest divisor of N' is larger than q.

Verifiable Secret Sharing (VSS). A standard secret sharing scheme is designed to be resilient against passive attacks. In many applications, a secret sharing scheme needs to be secure against the malicious dealer or parties with active attacks. This is achieved through VSS schemes, which were first introduced in 1985 [14]. Shamir secret sharing scheme by default does not qualify as a VSS scheme, as it does not provide protection against malicious participants (i.e., the dealer and shareholders).

2.3 Threshold Signatures

A threshold signature scheme enables a group of authorized parties to collectively sign a message m, generating a signature σ that can be verified using a single public key pk. Specifically, a threshold signature scheme, in terms of an $(t+1, n)$-threshold access structure, is defined as follows:

Definition 7. *A* threshold digital signature scheme *consists of three probabilistic algorithms:* KeyGen*,* Sign*, and* Verify*.*

- $\mathsf{KeyGen}\left(1^{\lambda}\right)$*: Given the security parameter as input and returns the public key* pk *along with a set of secret keys* sk_i *- one secret key per party. (For simplicity, we limit ourselves to the case where each party has a single share of the secret, focusing on Shamir and full-threshold secret sharing.)*
- $\mathsf{Sign}\left(\{\mathsf{sk}_i\}_{i\in Q}, m\right)$*: Given as input a* qualified set *of private keys and a message and returns a signature on the message.*
- $\mathsf{Verify}\left(\mathsf{pk}, (\sigma, m)\right)$*: Given* pk *and a signature* σ *on a message* m*, and outputs a bit that is equal to one if and only if the signature on* m *is valid.*

In essence, security for a threshold signature scheme means that an unqualified group of parties cannot forge a signature on a new message. In addition, for distributed signatures, we require that a valid output signature is indistinguishable from the signature produced by the signing algorithm of the underlying non-thresholdized scheme with the same public key.

3 VSS from ZK Proofs Over Shared Data

In this section, we propose a novel Non-Interactive Verifiable Secret Sharing (NI-VSS) scheme that utilizes ZK proofs over secret shared data [7, 10] to prove the validity and consistency of the individual shares. The proposed scheme does not rely on a concrete cryptographic hard problem, rather than a random oracle and a collapsing (quantum) computationally hiding commitment scheme.

To build the NI-VSS scheme, we first construct a non-interactive proof scheme which allows a single prover to convince a set of verifiers that they have each received a distinct evaluation of a polynomial $f(X) \in \mathbb{Z}_N[X]_t$.[4] It is worth noting that to achieve soundness, the number of honest verifiers is supposed to exceed t. On the other hand, to achieve threshold zero-knowledge, we assume that an adversarial prover can corrupt at most t verifiers. Thus, we assume the number of verifiers to be greater than or equal to $2t + 1$. We then demonstrate that our proposed scheme satisfies completeness (Def. 4), soundness against the prover and t malicious verifiers (Def. 5), and threshold ZK (Def. 6). We subsequently use the resulting Non-Interactive Threshold ZK (NI-TZK) proof scheme and build an efficient NI-VSS scheme based on Shamir secret sharing.

3.1 A NI-TZK Proof Protocol for Shamir Secret Sharing

As the key building block for our novel NI-VSS scheme, in this section, we present an efficient NI-TZK proof scheme that can be used to build a NI-VSS based on Shamir secret sharing. The new NI-TZK proof scheme is built for a collection of relations $R_1, \ldots, R_n$ with the same witness space, where each statement can be verified independently by individual verifiers. Given the shared input $x = x_1 \parallel x_2 \parallel ... \parallel x_n$, the prover proves the existence of a witness w that satisfies $(x_i, w) \in R_i$ for every $i \in 1, \ldots, n$. The proof includes proof pieces

[4] In general $\mathbb{Z}_N$ will constitute a ring. In later applications, we sometimes choose N to be a prime, so that $\mathbb{Z}_N$ becomes a field.

Prover: Given, a witness polynomial $f(X) \in \mathbb{Z}_N[X]_t$, an input $x = (x_1, \cdots, x_n)$, proceed as follows and output a proof π of the relations in eq. (3).

1. Sample $b(X) \leftarrow \mathbb{Z}_N[X]_t$ uniformly at random;
2. For $i = 1, \ldots, n$: Sample $y_i, y_i' \leftarrow \{0,1\}^\lambda$ uniformly at random; Set $\mathsf{C}_i \leftarrow \mathcal{C}(b(i), y_i)$ and $\mathsf{C}_i' \leftarrow \mathcal{C}(x_i, y_i')$;
3. Set $d \leftarrow \mathcal{H}(\mathsf{C}, \mathsf{C}')$, where $\mathsf{C} = (\mathsf{C}_1, \ldots, \mathsf{C}_n), \mathsf{C}' = (\mathsf{C}_1', \ldots, \mathsf{C}_n')$;
4. Set $r(X) \leftarrow b(X) - d \cdot f(X) \mod N$;
5. Set $\pi := (\mathsf{C}, \mathsf{C}', r(X), \{\pi_i\}_{i=1}^n)$, where $\pi_i = (y_i, y_i')$;
6. Publish $(\mathsf{C}, \mathsf{C}', r(X))$; Send individual proof $\{\pi_i = (y_i, y_i')\}_{i=1}^n$ to verifier V_i.

Verification: For $i = 1, \cdots, n$, each verifier (shareholder) i has a statement $x_i \in \mathbb{Z}_N$, and a proof $((\mathsf{C}, \mathsf{C}', r(X)), (y_i, y_i'))$. Given the set of statements and proofs for $i \in 1, \ldots, n$ the verifiers (i.e., shareholders) proceed as follows:

1. Verifier i acts as below and outputs `true` or `false`.
   (a) If $\mathsf{C}_i' \neq \mathcal{C}(x_i, y_i')$ return `false`;
   (b) Set $d \leftarrow \mathcal{H}(\mathsf{C}, \mathsf{C}')$;
   (c) If $\mathsf{C}_i == \mathcal{C}(r(i) + d \cdot x_i, y_i)$ return `true`; otherwise `false`;
2. Return `true` if *all* the verifiers return `true`; otherwise returns `false`.

Fig. 1. A NI-TZK Proof Scheme for Shamir Secret Sharing.

$\{\pi_i\}_{i \in 1,\ldots,n}$, where π_i allows the verifier V_i to check the validity of x_i in relation to R_i. The prover has a secret polynomial $f(X) \in \mathbb{Z}_N[X]_t$, and wants to prove the following n-distributed relations,

$$R_i = \{(x_i, f(X)) | f(i) = x_i\}, \tag{3}$$

where $i = 1, \ldots, n$. For the sake of convenience, we will refer to the relation mentioned above as the Shamir relation throughout the rest of the paper.

Fig. 1 describes the proof generation and verification of the new NI-TZK proof scheme for the Shamir relation given in equation (3), where $\mathcal{H} : \{0,1\}^* \to \Xi_k$ is a random oracle with Ξ_k an exceptional set of size k,[5] and $\mathcal{C} : \{0,1\}^* \times \{0,1\}^\lambda \to \{0,1\}^{2\lambda}$ is a commitment scheme that is collapsing [37, Def. 12] and quantum computationally hiding. Next, we show the proposed NI-TZK proof scheme (given in Fig. 1) satisfies the key security requirements of a ZK proof protocol over shared data, as defined in Sec. 2.1.

Remark 2. The challenge space of the protocol in Fig. 1 is $|\Xi_k| = k$. When $\mathbb{Z}_N$ is a cryptographically sized field, we can easily choose $\Xi_k = \mathbb{Z}_N$ to achieve a negligible soundness error, i.e. below $2^{-\lambda}$. In the case where $\mathbb{Z}_N$ is a ring, we might have the case that the largest exceptional set has size $k < 2^\lambda$. In that case the protocol from Fig. 1 would have to be repeated in the standard fashion: defining $S = \lceil \lambda / \log k \rceil$, we would have to sample S different $b_j(X)$ and construct S responses $r_j(X) = b_j(X) - d_j f(X)$, sampling the d_j using the hash function $\mathcal{H} : \{0,1\}^* \to (\Xi_k)^S$.

[5] Such a hash function can easily be implemented by hashing into a set $\{1, \ldots, k\}$ and then using the output value $i \in \{1, \ldots, k\}$ as an index in Ξ_k, i.e. $c_i \in \Xi_k$.

Theorem 1 (NI-TZK Proof Scheme for Shamir Secret Shares). *Let L be an n-distributed language for the list of relations given in equation (3), $t \geq 0$ be a security threshold such that $n \geq 2t+1$, and $h = n - t$. Assuming that the commitment scheme $\mathcal{C}$ is collapsing and quantum computationally hiding, for any potential set $I \subseteq [n]$ of size $|I| \geq h$, the protocol given in Fig. 1 is a non-interactive distributed proof scheme for L that satisfies completeness, threshold ZK, and soundness against the prover and t malicious verifiers in the QROM.*

Completeness. If the protocol is followed honestly and if the input was a valid statement-witness pair $(x, w) \in R$, where each verifier V_i, $1 \leq i \leq n$, has an input piece (share) x_i, then the verification will always accept the proof. Note that, the prover commits to the evaluations of $f(i)$ and $b(i)$ for $i = 1, \cdots, n$, and given x_i and $r(X) = b(X) - d \cdot f(X)$, the verifier V_i computes $r(i) + dx_i = b(i) - df(i) + dx_i = b(i)$, if $x_i = f(i)$. So if the witness is valid, then the commitments C_i and C'_i match and the verification (i.e., all the shareholders) will return `true` and accept the proof.

Soundness Against the Prover and t Malicious Verifiers. The NI-TZK scheme presented in Fig. 1 is made non-interactive using a variant of Fiat-Shamir transform which is proposed by Boneh et al. [10] for proofs on distributed data, and its security is also proven formally in [7] for a particular protocol. The commonly used transform of Fiat-Shamir [20], which is analysed in the (Quantum) Random Oracle model [18,38], is applied on a public coin interactive proof systems. Such that, instead of getting the challenge from the verifier, in the non-interactive protocol the prover applies a random oracle $\mathcal{H}$ to the concatenation of the input (i.e., the statement), and the communication transcript up to that point. In the case of sigma protocols the communication transcript is the commitment made in the first round. But in this variant of the Fiat-Shamir transform, the challenge is public, but the input (i.e., the statement) is shared among the verifiers and cannot be revealed to any single verifier. To deal with this concern, the idea is to generate the random challenge using the joint view of the verifiers in previous rounds [10]. Namely, the prover obtains the random challenge value as the hash of concatenation of n public commitments to the individual shares (i.e., shares of statement), and n public commitments produced in the initial round of the sigma protocol. Note that in this variant, each individual secret share is linked to a public commitment which satisfies (perfect) binding and (computational) hiding and can be verified by the corresponding shareholder.

The following Lemma is proven in [7], which proves the soundness of a NI-TZK argument that is built using the above variant of Fiat-Shamir transform.

Lemma 1. *Suppose $\Sigma = (P_1, V_1, P_2, V_2)$ is a sigma protocol for the relation R with super-polynomially sized challenge space $\mathcal{C}h$, special soundness, and quantum computationally unique responses. Let $\Sigma' = (P'_1, V'_1, P'_2, V'_2)$ be the following sigma protocol:*

$$P'_1(x, w) : y \leftarrow \{0,1\}^\lambda,\ \mathsf{C}_x \leftarrow \mathcal{C}(x, y),$$

$$
\begin{aligned}
com \leftarrow P_1(x,w),\ com' = (\mathsf{C}_x, com) \\
V_1'(com') : ch \leftarrow \mathcal{C}h \\
P_2'(ch) : rsp \leftarrow P_2(ch), rsp' \leftarrow (x,y,rsp) \\
V_2'(x,com',ch,rsp') : \textit{accept if } \mathsf{C}_x = \mathcal{C}(x,y) \textit{ and } V_2(x,com,ch,rsp) = 1
\end{aligned}
$$

Then the non-interactive version of Σ', transformed by the mentioned variant of Fiat-Shamir transform is a non-interactive quantum proof of knowledge for the same relation R, assuming that $\mathcal{C}$ is a collapsing commitment.

In the rest, we prove the protocol (given in Fig. 1) satisfies soundness against an adversarial prover and t malicious verifiers. Note that we structure our proof along the lines of [7, Theorem 2], but do this for a different relation, which has very different implications.

Lemma 2. *The proof system given in Fig. 1 constitutes a NI-TZK argument in the QROM for the list of relations of equation (3) if the deployed commitment scheme is collapsing.*

Proof. The results from Boneh et al. [10] show that in a NI-TZK proof scheme over secret shared data, the best combination of soundness and ZK that we can achieve is threshold zero-knowledge combined with soundness against prover and t malicious verifiers. To achieve this, we require to have at least $t+1$ honest parties among $n \geq 2t+1$ verifiers, i.e. be in the honest majority setting. Achieving these combinations means that in the target NI-TZK proof scheme, the prover can collude with t malicious verifiers to break the soundness, and at most t verifiers are allowed to collude to break the ZK and learn about the witness.

As a result, we need to prove that for any set $I \subset \{1, \cdots, n\}$ of honest parties where $|I| > t$ and any poly-time quantum adversary $\mathcal{A}^{RO}$, the following advantage is negligible:[6]

$$
\mathsf{Adv}^{\mathsf{sound}}_{\mathcal{A},I}(\lambda) = \Pr\left[\begin{matrix}\forall i \in I : V^{RO}(i, x_i, \widetilde{\pi}, \pi_i) = 1 \\ \nexists w \forall i : (x_i, w) \in R_i\end{matrix}\middle| \{(x_i, \pi_i)\}_{i\in I} \leftarrow \mathcal{A}^{RO}(1^\lambda)\right] .
$$

For compactness, we use the index I to denote the set of elements with index $i \in I$, e.g. $x_I = \{x_i\}_{i\in I}$. We further define the function F, which on the input of the data available to the set I, outputs the commitments as follows.

$$
\begin{aligned}
F : \Xi_k \times \mathbb{Z}_N^{|I|} \times \{0,1\}^{\lambda|I|} \times \mathbb{Z}_N[X]_{\leq t} \rightarrow \{0,1\}^{2\lambda} \\
(d, x_I, y_I, r(X)) \mapsto \{\mathcal{C}(r(i) + dx_i, y_i)\}_{i\in I} ,
\end{aligned}
$$

Let us define the following protocol $\Sigma' = (P_1', V_1', P_2', V_2')$.

$$
\begin{aligned}
P_1'(x_I, w) : \forall i \in I : y_i, y_i' \leftarrow \{0,1\}^\lambda,\ \mathsf{C}_i' \leftarrow \Xi(x_i, y_i') \\
b(X) \leftarrow \mathbb{Z}_N[X]_{\leq t},\ \mathsf{C}_I = F(0, x_I, y_I, b(X)) \\
V_1'(\mathsf{C}_I, \mathsf{C}_I') : d \leftarrow \Xi_k
\end{aligned}
$$

[6] For $|I| \leq t$, there always exists a witness that satisfies the relation.

$$P_2'(\mathbf{c}) : r(X) = b(X) - dw,\ rsp' = (y_I, y_I', r(X))$$
$$V_2'(rsp) : \text{accept if } \mathsf{C}_I' = \mathcal{C}(x_I, y_I) \text{ and } \mathsf{C}_I = F(d, x_I, y_I, r(X)).$$

It is clear that, if $F(RO(\mathsf{C}, \mathsf{C}'), x_I, y_I, r(X)) = \mathsf{C}_I$ and $\mathsf{C}_I' = \mathcal{C}(x_I, y_I')$, then $V^{RO}(I, x_I, \widetilde{\pi}, \pi_I) = 1$.[7] This implies that $\mathsf{Adv}^{\mathsf{sound}}_{\mathcal{A},I}(\lambda)$ is indeed negligible if the previously discussed variant of the Fiat-Shamir transform of Σ' is a (quantum) computationally sound proof of R_I as per [18, Definition 9]. This, in turn, is implied by proving that the transformed protocol is a quantum proof of knowledge as per [18, Definition 14]. We prove this last step by using Lemma 1, which implies the soundness of our protocol from Figure 1, when the following protocol has super-polynomially sized challenge space Ξ_k, special soundness, and quantum computationally unique responses.

$$\begin{aligned} P_1(x_I, w) : \forall i \in I : y_i &\leftarrow \{0,1\}^\lambda, \\ b(X) &\leftarrow \mathbb{Z}_N[X]_{\leq t}, \mathsf{C}_I = F(0, x_I, y_I, b(X)) \\ V_1(\mathsf{C}_I) : d &\leftarrow \Xi_k \\ P_2(\mathbf{c}) : r(X) = b(X) - dw,\ & rsp' \leftarrow (y_I, r(X)) \\ V_2(rsp) :\ \text{accept if } \mathsf{C}_I &= F(d, x_I, y_I, r(X)) \end{aligned}$$

We end this proof by discussing that these three properties are satisfied.

- **Challenge space**: In the case where $\mathbb{Z}_N$ is a field, the exceptional set Ξ_k is simply the field itself, which by definition has size superpolynomial in λ. Otherwise, the maximal size of Ξ_k is limited by the smallest divisor of N. In that case, as mentioned in Remark 2 we can amplify the challenge space size to above 2^λ by repeating the protocol $\lceil \lambda / \log k \rceil$ times.
- **Special Soundness**: Let $(x_I, C_I, d, r(X))$ and $(x_I, C_I, d', r'(X))$ with $d \neq d'$ be two accepting transcripts. Now, if for some $i \in I$, we have $r(i) + dx_i \neq r'(i) + d'x_i$, then we have found a collision in $\mathcal{C}$. Otherwise, we can compute a witness for x_I via $\frac{r(X) - r'(X)}{d' - d}$. Note that $d' - d$ is invertible because they are distinct elements from an exceptional set Ξ_k.
- **Unique responses**: Using the results from [7, Section A.2], this property is guaranteed in case $\mathcal{C}$ is collapsing and that $r(X)$ are unique. The latter follows from the fact that the function $(r, d, x) \mapsto r + dx$ is injective if d is an element from an exceptional set. □

Threshold zero-knowledge. We begin this section by stating the following Lemma.

Lemma 3. *The protocol in Fig. 1 satisfies the TZK property in the QROM for the list of relations of (3) if the used commitment scheme is quantum computationally hiding and collapsing, and if the underlying sigma protocol has honest-verifier zero-knowledge, completeness, and unpredictable commitments.*

We skip the proof of Lemma, as it immediately follows from the discussion in [7, Section A.3]. There, the authors show that it suffices to show zero-knowledge for

[7] Read component-wise, e.g. $\forall i \in I :\ \mathsf{C}_i' = \mathcal{C}(x_i, y_i')$.

any $I \subset \{0, \ldots, n\}$.[8] The case $i = 0$ is not relevant in this work and for the cases $i = 1, \ldots, n$, we can readily apply the results of their Lemmas 4 and 5 to our protocol. By definition, our commitment is quantum computationally hiding and collapsing. We can therefore finish the proof by showing that the sigma protocol underlying Fig. 1 is complete, HVZK, and has unpredictable commitments.

- **Completeness**: It follows from the completeness of our protocol.
- **HVZK**: We can define a simulator that samples $y_I, y'_I \leftarrow \{0,1\}^\lambda$, $r(X) \leftarrow \mathbb{Z}_N[X]_t$ and $d \leftarrow \Xi_k$ uniformly at random, then sets $C_I = F(d, x_I, y_i, r(X))$ and $C'_I = \mathcal{C}(x_I, y'_I)$. Since these sampled elements are also uniformly random in the real execution of the protocol, the transcripts are indistinguishable.
- **Unpredictable commitments**: By [38, Definition 4], unpredictable commitments imply that for every $(x_I, w) \in R_I$, finding two different commitments $(C_I^{(1)}, C_I'^{(1)})$ and $(C_I^{(2)}, C_I'^{(2)})$ that satisfy the probability

$$\Pr\left[(C_I^{(1)}, C_I'^{(1)}) = (C_I^{(2)}, C_I'^{(2)}) \middle| \begin{matrix}(C_I^{(1)}, C_I'^{(1)}) \leftarrow P_1(x_I, w) \\ (C_I^{(2)}, C_I'^{(2)}) \leftarrow P_1(x_I, w)\end{matrix}\right]$$

  is negligible in the security parameter λ. There are two options to get such a collision, either the inputs to $\mathcal{C}$ are equal, or we find a collision in $\mathcal{C}$. The former happens with negligible probability, since the inputs to $\mathcal{C}$ are uniformly distributed in $\mathbb{Z}_N \times \{0,1\}^\lambda$ and the latter is prevented by the fact that $\mathcal{C}$ is collapsing. □

This completes the proof of Theorem 1.

3.2 A NI-VSS Scheme from NI-TZK Proofs

Next, we use the NI-TZK proof scheme proposed in the last subsection and construct a NI-VSS scheme based on Shamir secret sharing. Our scheme operates on the assumption that each shareholder has a secure communication channel with the dealer, which can be achieved through a public key infrastructure. Therefore, the shares will only be hidden computationally. The proposed scheme works in the majority honest setting, and the validity of secret shares cannot be publicly verified (as in [4,31]), and it requires (non-interactive) collaboration among the shareholders to verify them. We demonstrate later that this is sufficient in many Shamir-based threshold protocols (e.g. DKGs, threshold signatures, etc.) that also work in the majority-honest setting.

Our Definitions. Before going through the proposed construction to build a NI-VSS scheme, we review our formal definitions of VSS schemes which are a minimally modified version of the ones from previous works [28,31].

Definition 8. *An (n, t, x_0) non-interactive VSS consists of four PPT Algorithms of (Initialization, Share, Verification, Reconstruction) as follows:*

[8] Actually, the authors implicitly prove TZK, but do not call it as such.

1. *Initialization: In this phase, the system parameters are generated and shared with the parties.*
2. *Share*$(n, t, x_0) \to (x_1, \cdots, x_n, \pi)$*: Given the number of parties* n*, threshold* t*, and the secret* x_0*, the algorithm secret shares* x_0 *and outputs the shares* $\{x_1, \cdots, x_n\}$ *and a proof* π *to prove that it has done the sharing correctly.*
3. *Verification*$(n, t, x_1, \cdots, x_n, \pi) \to$ `true/false`*: Given the number of parties* n*, threshold* t*, and the shares* $x_1, \cdots, x_n$ *(or encryption of them), and the proof* π*, generated by Share, the algorithm outputs either* `true` *or* `false`*.*
4. *Reconstruction*$(n, t, x_1, \cdots, x_{t+1}) \to x_0/\{\mathsf{true/false}\}$*: Given any* $t+1$ *of the shares, e.g.,* $\{x_1, \cdots, x_{t+1}\}$*, it reconstructs and returns* x_0*. Alternatively, given a candidate value for* x_0 *(or in general a function of it) and* t *(or in general* $t+1$*) of the shares, the algorithm confirms the validity of the candidate secret* x_0 *(or the function of it), and returns either* $\{\mathsf{true/false}\}$*.*

A verifiable secret sharing scheme further has two requirements as follows [31].

- **Verifiability constraint**: A shareholder must be able to determine whether a share of the secret is valid or not. If it is valid, then Reconstruction should produce a unique secret x_0 when run on any $t+1$ distinct valid shares. Alternatively any t (or in general $t+1$) shareholders should be able to check the validity of a potential value of x_0 (or in general a function of it).
- **Unpredictability**: The protocol must be unpredictable, meaning that there is no strategy for selecting t shares of the secret that would enable someone to predict the secret x_0 with a significant advantage.

We highlight that our definition of VSS differs slightly from current ones [28, 31]. Our definitions utilize a ZK proof scheme over secret shared data for proving the validity of the shares, ensures the existence of a polynomial-time verification algorithm that can validate the shares, and also introduces a *novel approach* for reconstruction of the main secret. The first two features already are implicitly built in any VSS scheme, however the third one is new in our framework. In our Reconstruction algorithm, in addition to enabling the reconstruction of the secret x_0 using Largange interpolation by any $t+1$ shareholders, we also consider a scenario where the dealer disclose a candidate value for x_0 (or in general a function of it) and t (or in general $t+1$) of the shareholders can validate the validity of the disclosed secret (or the correctness of a computation performed on x_0). Later, we demonstrate that the new reconstruction approach is commonly used in practice, and shareholders typically do not reconstruct the plain value of x_0. Instead, each shareholder acts as a dealer once and subsequently employs their shared secret to perform certain computations. They then provide a ZK proof to prove the correctness of their actions. In some cases, these proofs may only be verifiable by the shareholders themselves.

Our Construction. In a VSS scheme, a dealer aims to distribute shares of a secret x_0 among n parties $P_1, \ldots, P_n$. Such that depending on the underlying access structure, a subset of shareholders are qualified to recover the secret x_0. In our case, which is based on Shamir secret sharing, the secret can be recovered by any

subset of more than t shareholders, where $t < n$. On the other hand, any subset of size $\leq t$ will not gain any information about x_0, unless the security of underlying NI-TZK proof scheme is broken. The complexity of our VSS scheme is linear in the security parameter and also linear in the number of shareholders which is essentially optimal, but notably our scheme only uses lightweight operations (such as hashing and polynomial evaluations). It achieves computational security, which is proven in the (Q)ROM, using a secure commitment scheme. We present our protocol in Fig. 2.

It is important to note that in the *Reconstruction* with the new approach, unlike the Lagrange interpolation based approaches, the parties do not perform any decryption or proof generation to show the correctness of their actions. Instead, the dealer calculates and publishes the reconstructed secret value $f(0) = x_0$ (or a function thereof) along with a NI-TZK proof for the distributed relation $R_i = \{(x_i, f(X))|f(i) = x_i\}$, for $i = 0, 1, \ldots, n$. Then, any t (or in general $t + 1$ if a function of x_0 is reconstructed) shareholders use their secret shares to verify the validity of the disclosed secret value x_0 (or a function of it). If the verification process returns `true`, this confirms that the disclosed value x_0 (or a function of it) represents the main secret $f(0)$ (or a function thereof). Note that as in other schemes, if we have $t+1$ shares, we only can achieve a reconstruction with *abort*, while with n shares we can have a *robust* reconstruction phase.

It's important to highlight that, similar to the first reconstruction approach, where any subset of qualified sets can recover the secret $f(0)$ and fewer than a threshold number of them gain no knowledge about $f(0)$, the new reconstruction approach also allows only a qualified subset of the shareholders to confirm the soundness of the underlying NI-TZK proof scheme and the validity of the disclosed value x_0. Conversely, relaying on the TZK property of the underlying NI-TZK proof scheme, any group of parties that is smaller than the threshold value will be unable to confirm the authenticity of the revealed value x_0 (or any function derived from it) and gain information about the original secret $f(0)$ (or any computation involving the original secret $f(0)$).

In practical distributed protocols, the dealer typically does not reveal the main secret $f(0)$. Instead, they disclose the results of specific computations carried out with it, such as $h_0 = g^{f(0)}$ in the context of distributed generation of a DL tuple. In these cases, the dealer is required to publish a proof that the computation was conducted using $f(0)$, e.g., $h_0 = g^{f(0)} \wedge f(i) = x_i$ in the context of distributed key generation for signature schemes such as Schnorr and BLS. At first glance, this may seem unconventional. However, as we will show later, it is actually sufficient and common practice in many threshold protocols, e.g., DKGs and threshold signatures. In the upcoming section, we will explore some applications and types of NI-TZK proof systems that one might need in the *Reconstruction* phase.

Theorem 2 (VSS from NI-TZK Proof Schemes). *If the proof scheme given in Fig. 1 is a secure NI-TZK protocol for the relations in equation (3), then, the non-interactive VSS scheme (given in Fig. 2) is secure. That is, (i) the Reconstruction protocol results in the secret distributed by the dealer for any*

Initialization: Parties $P_1, \cdots, P_n$ generate system parameters and each one registers a PK to facilitate secure communications.

Share: Given n and t, to share x_0, the dealer proceeds as follows:

1. Sample a uniformly random polynomial $f(X)$ of degree t with coefficients in a ring R, subject to $f(0) = x_0$.
2. For $i = 1, 2, \cdots, n$: set $x_i := f(i)$.
3. Given $f(X)$ and $x = (x_1, \cdots, x_n)$, run the prover of NI-TZK scheme in Fig. 1, and obtain the proof $\pi := (\mathsf{C}, \mathsf{C}', r(X), \{\pi_i\}_{i=1}^n)$.
4. Send the share and the individual proof (x_i, π_i) privately to party P_i and broadcast the elements $(\mathsf{C}, \mathsf{C}', r(X))$ as the proof.

Verification: To verify the received shares, $P_1, \cdots, P_n$ utilize their shares $\{x_i\}_{i=1}^n$ and run the verifier of the NI-TZK proof scheme given of Fig. 1. If the verification of P_i fails, then P_i broadcasts a complain against the dealer. If more than t shareholders complain against the dealer, then the *Verification* returns `false`. If P_i complains that his part of proof does not verify, the dealer broadcasts $(x_i, \pi_i := (y_i, y_i'))$ so that everyone can verify it using the verification algorithm of the NI-TZK scheme. If it passed the verification, the protocol continues as normal, otherwise the parties disqualify the dealer and *Verification* returns `false`. Since disqualifying the dealer or parties happens on the basis of only broadcasted information, at the end all the honest shareholders will agree on the same set of qualified parties $Q \subseteq \{1, 2, \cdots, n\}$ or will reject the final verification. At the end, if the verification returns `true`, all honest shareholders are sure that they have received a valid share of $x_0 = f(0)$, and any subset of size larger than t of them can retrieve the secret x_0.

Reconstruction: This can be done through two approaches: either by using Lagrange interpolation as in previous works or by employing a novel approach outlined below. In the new approach, the dealer reconstructs (i.e., reveals) the secret x_0 and also proves its validity, and for that the process proceeds as follows:

1. Given the witness $f(X)$, the dealer computes (reconstructs) $x_0 = f(0)$.
2. Using $f(X)$ and $x = (x_0, x_1, \cdots, x_n)$, run the prover of the NI-TZK scheme in Fig. 1 for $i = 0, 1, \ldots, n$, to prove that $f(0) = x_0 \wedge f(i) = x_i$ for $i = 1, \ldots, n$, and obtain $(x_0, y_0, y_0', \{\mathsf{C}_i, \mathsf{C}_i'\}_{i=0}^n, r(X), \{\pi_i\}_{i=1}^n)$. This allows the dealer to convince the shareholders that their shares come from a polynomial of degree t with free term $x_0 = f(0)$.
3. Send the individual proof $\pi_i := (y_i, y_i')$ privately to party P_i, and broadcast the elements $(x_0, y_0, y_0', \{\mathsf{C}_i, \mathsf{C}_i'\}_{i=0}^n, r(X))$.
4. Each shareholder P_i has secret values $(x_i, \pi_i := (y_i, y_i'))$, and a public proof $(x_0, y_0, y_0', \{\mathsf{C}_i, \mathsf{C}_i'\}_{i=0}^n, r(X))$. Given the set of statements and proofs, the shareholders run the verification of the NI-TZK scheme in Fig. 1 and return either `true` or `false`. Note that in this case, each shareholder P_i additionally checks if $\mathsf{C}_0' = \mathcal{C}(x_0, y_0') \wedge \mathsf{C}_0 == \mathcal{C}(r(0) + d \cdot x_0, y_0)$.
5. At the end, the algorithm **return** `true` if *all* the shareholders return `true`; otherwise **return** `false`. Returning `true`, confirms that the value x_0 is the reconstruction of the main secret value $f(0)$.

Fig. 2. The proposed NI-VSS scheme.

qualified set of shareholders, (ii) any non-qualified set of shareholders is unable to recover the secret (i.e., unpredictability).

Proof. In Theorem 1, we showed that the protocol presented in Fig. 1 is a NI-TZK scheme for L, that is an n-distributed language for the list of relations given in equation (3), satisfies completeness, threshold ZK, and soundness against the prover and t malicious verifiers in the QROM.

Completeness of the NI-TZK scheme implies that if the parties $P_1, \cdots, P_n$ follow the protocol, then at the end of the *Sharing* phase, each of them obtain a distinct evaluation of a polynomial degree t, where $t < n$. Relaying on the fact that any degree t polynomial is uniquely determined by $t+1$ distinct evaluations, any $t+1$ of n shareholders can use Lagrange interpolation and reconstruct $f(X)$ and retrieve the value $f(0)$, which is the secret value in the NI-VSS scheme. Similarly, any t (or in general any $t+1$) shareholders can also verify the proof given by dealer in the *Reconstruction* phase, and ensure that the secret value x_0 revealed by the dealer, is equal to the secret shared value $f(0)$.

The NI-TZK scheme's soundness against prover and t malicious verifiers implies that if there is no n-distributed input $x' \in L_R$ such that $x_i = x'_i$, for all honest parties P_i, then the protocol (honest verifiers) will reject the proof except with negligible probability. Therefore, a malicious dealer would have to either break the soundness of the underlying NI-TZK proof scheme or it will be caught with an overwhelming probability. It is important to note that, during the verification process any conflicts between the dealer and shareholders are resolved using the method outlined in the Verification algorithm. In the scenario where the majority of shareholders are honest, this enables the parties to achieve robustness within the resulting NI-VSS scheme.

For unpredictability, the threshold ZK property of the underlying NI-TZK scheme guarantees that any polynomial-time adversary $\mathcal{A}$ that controls up to t verifiers cannot learn anything about the secret polynomial $f(x)$, including the value $f(0)$. As a result, any non-qualified set of shareholders is unable to recover the secret $x_0 = f(0)$. In other words, if an adversary who controls a non-qualified set of shareholders can learn about the secret value $f(0)$, they can be used as an adversary against the threshold ZK property of the NI-TZK proof scheme. □

3.3 Asymptotic Costs and Empirical Performance

Next, we first summarize the efficiency metrics for our proposed NI-VSS scheme, and then assess its empirical performance through a prototype implementation. To gauge the efficiency of new scheme, we also conduct a comparative analysis with the widely used VSS scheme of Pedersen [28].

Asymptotic Costs. As in Shamir secret sharing, to share a secret x_0 among n parties with threshold value t, the dealer first computes n evaluations of a degree-t polynomial $f(X)$. Then, it runs the prover of NI-TZK scheme outlined in Fig. 1 and generates a proof for the correctness of the shearing phase. To this end, the dealer needs to compute n evaluations of a new degree-t polynomial $b(X)$, compute $2n$ commitments, query one time to the RO, and perform t subtractions between the coefficients of $f(X)$ and $b(X)$. To verify their shares, the

shareholders participate in the verification of the NI-TZK proof system (outlined in Fig. 1) and disseminate the final output to the network. As part of this process, each shareholder must compute two commitments, one evaluation of a random oracle, one polynomial evaluation of degree t, and one addition over $\mathbb{Z}_N$. In terms of communication, the dealer broadcasts $(\mathsf{C}, \mathsf{C}', r(X))$ to the network, which consists of $2n$ commitments and t polynomial coefficients. It also sends the individual proof (x_i, π_i) privately to party P_i, which consists of 3 $\mathbb{Z}_N$ elements.

In Pedersen VSS scheme, to share a secret, a dealer needs to evaluate a degree-t polynomial $2n$ times (i.e., n times with $f(X)$ and n times with $b(X)$), compute n exponentiations and $t \approx n/2$ multiplications in the underlying group $\mathbb{G}$. Then, the dealer needs to broadcast $t \approx n/2$ group elements as the commitments, and also privately send 2 field elements to each party P_i, as their shares. Then, to verify the shares, each verifier needs to compute $t \approx n/2$ exponentiations in the group $\mathbb{G}$. Table 2, summarizes the asymptotic costs of our proposed protocol and compares it with Pedersen's scheme [28]. As a crucial optimization in our scheme, we eliminate the need for additional randomness, represented as y_i and y_i' within the hashes (i.e., the commitments). This optimization results in shorter private communication from the dealer to the parties in the table.

Table 2. Asymptotic costs in Pedersen [28] and our proposed NI-VSS schemes. DL: Discrete Logarithm, BC: Broadcast, n: Number of parties, $E_{\mathbb{G}}$: Exponentiation in group $\mathbb{G}$, $M_{\mathbb{G}}$: Multiplication in group $\mathbb{G}$, $\mathcal{PE}$: degree-t Polynomial Evaluation, $\mathcal{H}$: Hashing, $|\mathbb{G}|$: $\mathbb{G}$ element size, $|\mathbb{Z}_q|$: $\mathbb{Z}_q$ element size, $|\mathbb{Z}_N|$: $\mathbb{Z}_N$ element size, $|\mathcal{H}|$: Output size of $\mathcal{H}$, DV: Designated Verifier, FS: Fiat-Shamir.

VSS Scheme	Assumption	Sharing	Dealer's Communication	Verification (DV)
Pedersen [28]	DL-based (IT & Classic)	$1n$ $E_{\mathbb{G}}$ $2n$ $\mathcal{PE}$	Private: $2n$ $\lvert\mathbb{Z}_q\rvert$ BC: $t \approx 0.5n$ $\lvert\mathbb{G}\rvert$	$t \approx 0.5n$ $E_{\mathbb{G}}$
This work Sec. 3.2	Hash-based (PQ)	$2n$ $\mathcal{H}$ $2n$ $\mathcal{PE}$	Private: $1n$ $\lvert\mathbb{Z}_N\rvert$ BC: $2n$ $\lvert\mathcal{H}\rvert$ + $0.5n$ $\lvert\mathbb{Z}_N\rvert$	1 $\mathcal{PE}$ + 3 $\mathcal{H}$ (1 $\mathcal{H}$ is for FS)

Empirical Performance. To assess the practical performance of the new VSS scheme, we implemented a prototype of it alongside the Pedersen scheme using SageMath. In the new VSS scheme, we employed a SHA256 hash function for instantiating the commitment scheme $\mathcal{C}$ and the random oracle $\mathcal{H}$. For the Pedersen scheme implementation, we utilized `Curve 25519`, and optimized the implementation through Montgomery x-arithmetic. Additionally, we relied on SageMath's built-in functions for handling polynomials and hash operations.

To evaluate their performance, we conducted experiments where we varied the number of parties and the threshold value. Specifically, we report the run times of the *Sharing* and *Verification* phases, along with the communication size for different numbers of parties, i.e., n, and threshold values, i.e., t. To conduct these experiments, we ran our code on a laptop with Ubuntu 22.04 LTS, a 11th Gen Intel(R) Core(TM) i9-11950H at base frequency 2.60GHz, and 64GB of memory. All the operations in the sharing and verification phases are done in a single-thread mode. The performance results with different values of (n, t), ranging from $(32, 15)$ to $(16384, 8191)$ are summarized in Table 3.

Table 3. Empirical performance of NI-VSS schemes Pedersen [28] and our proposed scheme for various numbers of parties and threshold values (n, t). n: Number of parties, t: Threshold value, BC: Broadcast, $|\mathbb{Z}_q| = |\mathbb{Z}_N| = |\mathbb{G}| = |\mathcal{H}| = 256$ bits.

(n, t)	Scheme	Sharing	Dealer's Communication	Verification
$(32, 15)$	Pedersen [28]	78.1 msec	Private: 2.0 KB + BC: 0.5 KB	10.2 msec
	This Work	2.40 msec	Private: 1.0 KB + BC: 2.5 KB	0.12 msec
$(128, 63)$	Pedersen [28]	310 msec	Private: 8.0 KB + BC: 2.0 KB	97.6 msec
	This Work	13.7 msec	Private: 4.0 KB + BC: 10.0 KB	0.36 msec
$(512, 255)$	Pedersen [28]	1.310 sec	Private: 32.0 KB + BC: 8.0 KB	525 msec
	This Work	0.140 sec	Private: 16.0 KB + BC: 40.0 KB	1.20 msec
$(2048, 1023)$	Pedersen [28]	6.53 sec	Private: 128 KB + BC: 32 KB	2.35 sec
	This Work	2.00 sec	Private: 64 KB + BC: 160 KB	4.90 msec
$(8192, 4095)$	Pedersen [28]	48.1 sec	Private: 512 KB + BC: 128 KB	9.47 sec
	This Work	30.6 sec	Private: 256 KB + BC: 640 KB	0.019 sec
$(16384, 8191)$	Pedersen [28]	153.6 sec	Private: 1024 KB + BC: 256 KB	19.2 sec
	This Work	121.7 sec	Private: 512 KB + BC: 1280 KB	0.039 sec

The implementation results solidify the advantage of our NI-VSS scheme over the Pedersen scheme in both sharing and verification phases, in addition to post-quantum security. Specifically, our NI-VSS scheme demonstrates a remarkable speedup in the verification phase, achieving 271×, 437×, and 498× faster verification times than the Pedersen scheme for (n, t) equal to $(128, 63)$, $(512, 255)$, and $(8194, 4095)$, respectively. Similarly, in the sharing phase, for the same settings, our scheme is approximately 22.6×, 9.3×, and 1.57× faster than the Pedersen scheme. In terms of communication costs, our scheme has a broadcast cost for the dealer that is five times higher than the Pedersen scheme, but the private communication size is halved. These implementation results also affirm the practicality and scalability of the new NI-VSS scheme for deployment in various threshold protocols. One notable factor contributing to our improvements in the sharing and verification phases is using lightweight cryptographic operations such as hashing and polynomial evaluation, in the underlying NI-TZK proof scheme.

We highlight that our implementation remains relatively naive, operating in a single-threaded fashion without specific optimizations. A potential optimization strategy could involve adopting algorithms from [36], which offer improved computational complexity for evaluating a polynomial at multiple points, thus improving the efficiency of the sharing (and also verification in some cases).

4 More Efficient Threshold Protocols in the DL Setting

In this section, we leverage our new VSS scheme from Section 3 and revisit the well-known Pedersen DKG protocol [27] along with the threshold signature of Gennaro, Jarecki, Krawczyk, and Rabin [21], which uses Schnorr's signature [30] for signing and Pedersen's DKG protocol for generating the (ephemeral) keys.

4.1 An Efficient DKG Protocol for DL

Pedersen DKG protocol [27] allows a group of parties to generate a DL instance, e.g., $\mathsf{pk} = g^{\mathsf{sk}}$, in a fully distributed manner, where g is the generator of the DL group, and $(\mathsf{sk}, \mathsf{pk})$ are a pair of secret and public keys, respectively.

In the following, we present an efficient robust DKG protocol for distributed generation of a DL instance, which can outperform Pedersen's protocol. To this end, we first construct an efficient NI-TZK proof scheme for the DL problem that acts as a building block in our proposed DKG protocol (and threshold signature). The NI-TZK proof scheme allows a prover to convince a set of verifiers (i.e., shareholders) that $h = g^{f(0)} \wedge f(i) = x_i$, for the shared input $x = x_1 \parallel x_2 \parallel \cdots \parallel x_n$, a secret polynomial $f(X) \in \mathbb{F}_p[X]_t$ and a secret input $x_0 = f(0)$. One usually can face with a similar scenario in the DL-based threshold protocols (e.g., threshold variants of El Gamal, ECDSA, etc.). The new NI-TZK proof scheme is built for the following n-distributed relations,

$$R_i = \{(g, h, x_i, f(X)) | h = g^{f(0)} \wedge f(i) = x_i\}, \tag{4}$$

where $i = 1, \ldots, n$. Fig. 3 describes the algorithms of our proposed NI-TZK proof scheme for the DL relation, where $\mathcal{H}$ is a random oracle and $\mathcal{C}$ is a computationally hiding commitment scheme. Roughly speaking, the protocol is obtained by slightly modifying the conjunction of the Schnorr ID protocol with the NI-TZK scheme presented in Fig. 1. This is another instance of different NI-TZK proof schemes that one may need in the *Reconstruction* phase of the new VSS scheme, where parties reconstruct a function of the main secret $f(0)$, namely $h = g^{f(0)}$, rather than the plain value of it.

Prover: Given, $f(X) \in \mathbb{F}_p[X]_t$, and the input $x = (g, h := g^{f(0)}, x_1, \cdots, x_n)$, proceed as follows and output a proof π of the relations in equation (4).

1. Sample $b(X) \leftarrow \mathbb{F}_p[X]_t$ uniformly at random;
2. For $i = 1, \ldots, n$: Sample $y_i, y_i' \leftarrow \{0,1\}^\lambda$ uniformly at random and set $\mathsf{C}_i = \mathcal{C}(b(i), y_i)$ and $\mathsf{C}_i' = \mathcal{C}(x_i, y_i')$;
3. Sample $y_0, y_0' \leftarrow \{0,1\}^\lambda$ and set $\mathsf{C}_0 = \mathcal{C}(g^{b(0)}, y_0), \mathsf{C}_0' = \mathcal{C}(g \| h, y_0')$
4. Set $d \leftarrow \mathcal{H}(\mathsf{C}, \mathsf{C}')$, where $\mathsf{C} = (\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_n), \mathsf{C}' = (\mathsf{C}_0', \mathsf{C}_1', \ldots, \mathsf{C}_n')$;
5. Set $r(X) \leftarrow b(X) - d \cdot f(X) \mod p$;
6. Return $\pi := (g, h, \mathsf{C}, \mathsf{C}', r(X), y_0, y_0', \{\pi_i\}_{i=1}^n)$, where $\pi_i = (y_i, y_i')$;

Verification: Given $\pi := (g, h, \mathsf{C}, \mathsf{C}', r(X), y_0, y_0', \{\pi_i := (y_i, y_i')\}_{i=1}^n)$, the verifiers $\{V_i\}_{i=1}^n$ use their shares and individual proofs (x_i, π_i), and the verification proceeds as follows:

1. Verifier i acts as below and outputs `true` or `false`.
   (a) If $\mathsf{C}_i' \neq \mathcal{C}(x_i, y_i')$ or $\mathsf{C}_0' \neq \mathcal{C}(g \| h, y_0')$ return `false`
   (b) Set $d' \leftarrow \mathcal{H}(\mathsf{C}, \mathsf{C}')$;
   (c) Compute $h' = g^{r(0)} \cdot h^d$.
   (d) If $\mathsf{C}_i == \mathcal{C}(r(i) + d \cdot x_i, y_i) \wedge \mathsf{C}_0 == \mathcal{C}(h', y_0)$ return `true`; otherwise `false`;
2. Return `true` if *all* the verifiers return `true`; otherwise returns `false`.

Fig. 3. A NI-TZK proof scheme for discrete logarithm.

Round 1 (VSS and Committing): Each party P_i proceed as follows:

1. Sample $f^{(i)}(X) \leftarrow \mathbb{F}_p[X]_t$ subject to $f^{(i)}(0) = x_i$ and set $x_{ij} = f^{(i)}(j)$
2. Using $(g, h_i := g^{x_i}, \{x_{ij}\}_{j=1}^n)$, run the prover of NI-TZK proof scheme in Fig. 3, and obtain $\pi := (g, h_i, \mathsf{C}, \mathsf{C}', r(X), y_{i0}, y'_{i0}, \{\pi_{ij}\}_{j=1}^n)$, where $\pi_{ij} = (y_{ij}, y'_{ij})$.
3. Publish $(g, \mathsf{C}, \mathsf{C}', r(X))$ and store $(h_i, y_{i0}, y'_{i0}, \{x_{ij}, \pi_{ij}\}_{j=1}^n)$ for next round.

Round 2 (Opening, Verification, PK Computation):

1) **Opening**: Each party $\{P_i\}_{i=1}^n$ broadcasts $(h_i = g^{x_i}, y_{i0}, y'_{i0})$ and sends (x_{ij}, π_{ij}) privately to party P_j. If a party refuses to open a commitment, then that party is disqualified.
2) **Verification**: Each party $\{P_j\}_{j=1}^n$ uses $(\pi_{ij}, y_{i0}, y'_{i0})$ and verifies the correctness of the share x_{ij} it got from P_i with respect to $g, h_i = g^{x_i}$ for $i \neq j$, by running the verifier of NI-TZK proof scheme given in Fig. 1. If the verification fails, then P_j broadcasts a complaint against P_i. Any player with at least $t+1$ complaints is disqualified. If P_j complains that P_i's proof does not verify, then P_i broadcasts $(x_{ij}, y_{ij}, y'_{ij})$, so that everyone can verify it using the verification algorithm of NI-TZK scheme. If this verification succeeds, the protocol continues as normal, otherwise P_i is disqualified. Since disqualifying the parties happens on the basis of only broadcasted information, at the end, all parties will agree on the same set of qualified parties $Q \subseteq \{1, \ldots, n\}$ such that $x = \sum_{i \in Q} L_{0,i}^Q x_i$, where $L_{0,i}^Q = \prod_{j \in Q,\ j \neq i} \frac{j}{j-i}$ is a Lagrange coefficient.
3) **PK Computation:** Parties compute the public key as $g^x = \prod_{i \in Q} g^{L_{0,i}^Q x_i}$.

Fig. 4. Designated verifier DKG protocol for DL-based schemes.

Theorem 3 (NI-TZK Proofs for DL). *Let L be an n-distributed language for the list of relations given in equation (4), $t \geq 1$ be a security threshold such that $n \geq 2t+1$. Assuming that the commitment scheme $\mathcal{C}$ is computationally hiding, for any potential set $I \subseteq [n]$ of size $|I| \geq n - t$, the protocol described in Fig. 3 is a non-interactive distributed threshold ZK protocol for L that satisfies completeness, threshold ZK, and soundness against the prover and t malicious verifiers in the ROM.*

Proof. The proof is analogous to the proof of Theorem 1 which is omitted. We highlight that in this case, in the soundness proof, one reduces the security of scheme to the DL problem, thus this scheme only achieves classical security. □

Now, we can use the general NI-VSS of Fig. 2 and the NI-ZSK proof scheme given in Fig. 3, and construct a DKG protocol with designated verifiers for DL. The resulting DKG is described in Fig. 5 and can be considered as an adaption of the Pedersen DKG protocol version from [21] to work with NI-TZK proofs and the new VSS scheme.

Theorem 4. *Under the DL assumption, the protocol in Fig. 4 is a secure DKG protocol, namely it satisfies the correctness and secrecy properties against a malicious adversary corrupting up to t parties, with $t < n/2$.*

Proof. The proof is analogous to the ones in [7,21], but in this case the simulator of DKG scheme runs the simulator of NI-TZK proof scheme as a subroutine. □

KeyGen: Parties run the DKG protocol of Fig. 4. At the end, each party holds a verified secret share x_i of x. The resulting public key is g^x. We assume $Q_0 = \{1, \ldots, n\}$ to be the qualified set at the end of this step.

Sign$(m, \langle x_i \rangle)$**:** To sign a message m, the parties in Q_0 act as follows.

1. Parties run the DKG protocol of Fig. 4 to compute g^b. Malicious parties might be disqualified, so we end up with a set $Q_1 \subseteq Q_0$. Only the parties in Q_1 continue the protocol. Each party P_i in Q_1 holds a share b_i of b.
2. The parties compute the challenge $d \leftarrow H(g^b||m)$.
3. Parties in Q_1 behave as follows.
   (a) Each party P_i computes and broadcasts $r^{(i)}(X) = b^{(i)}(X) - ds^{(i)}(X)$.
   (b) Using the shared values from VSS phase, each other party P_j verifies
   $$r^{(i)}(j) \stackrel{?}{=} b^{(i)}(j) - ds^{(i)}(j)\,.$$
   (c) Whenever one of these checks fails, P_j broadcasts a complaint against P_i. When a player P_i has $t+1$ or more complaints against them, they are disqualified. The remaining players can then construct $r^{(i)}(0)$ by reconstructing both $b^{(i)}(0)$ and $s^{(i)}(0)$. This is always possible when there are at least $t+1$ honest parties.
   (d) For each party P_i in $Q = Q_0 \backslash Q_1$, set and reconstruct $r_l^{(i)}(0) = s_{d_l}^{(i)}(0)$.
   (e) Using $\{r^{(i)}(0)\}_{i=1,\ldots,n}$, parties build the response $r = \sum_{i \in Q} r^{(i)}(0)\,.$
4. Finally, parties output the signature $((r, d), m)$.

Verify$((r, d), m, \mathsf{pk})$**:** To verify a signature (r, d) on m using the public key $\mathsf{pk} = g^x$, the verifier proceeds as follows.

1. Compute $h = g^r \cdot (g^x)^{-d}$.
2. Compute $d' \leftarrow H(h||m)$.
3. If $d = d'$, return valid, otherwise invalid.

Fig. 5. A novel robust threshold signature scheme based on GJKR scheme [21].

4.2 More Efficient Threshold Signatures from Schnorr's Scheme

Next, using the DKG protocol given in Fig. 4, we modify the Schnorr-based threshold signature scheme of Gennaro, Jarecki, Krawczyk, and Rabin [21], and present a new variant of it that can be more efficient in practice. Fig. 5 represents the description of our proposed robust threshold signature scheme that uses Fig. 4 for the DKG and the distributed generation of the ephemeral key g^b.

Theorem 5. *Under the DL problem the threshold signature described in Fig. 5, is secure against a static adversary corrupting up to t parties, with* $t < n/2$.

We refer to App. A, for the security proof of the scheme.

Security Against Wagner's Attack. The threshold signature in Fig. 5 is secure against the concurrent attack using Wagner's algorithm [5, 40]. Intuitively, the attack crucially relies on the fact that an adversary can open ℓ sessions of the protocol in parallel, that is in the same round. At each session r the adversary gets g^{b_i} from the honest parties and computes its commitment shares g^{b_j}, $j \in A$, based on the honest parties' commitment shares, before submitting g^b to the RO. For big enough ℓ this is enough for the adversary to forge a signature [5]. As

mentioned in the same paper [5], a countermeasure is to let parties commit to the shares g^{b_i} and only after a round of broadcast they open them. This clearly prevents the adversary to compute his shares adaptively. A typical way for implementing this is using PK-based commitments such as Pedersen's, as done for example in [21, 26]. We have a similar approach in our protocol, and reveal the commitments and opening at the beginning of the second round.

4.3 Efficiency of New Protocols

Next, we summarize the efficiency of the proposed DKG protocol (Fig. 4) and the threshold signature (Fig. 5) and compare them with the ones proposed by Pedersen [28] and Gennaro et al. [21].

Comparing our DKG protocol to the variant of Pedersen DKG presented in [21], we observe asymptotic improvements in computational cost of parties. In our DKG protocol, each party is required to perform approximately $2n$ exponentiations within the group, conduct $3n$ evaluations of degree-t polynomials in the field, and execute $5n$ hash operations for commitment purposes. While, in the Pedersen DKG each party needs to compute $(2tn+n)$ exponentiations in the group, $2n$ degree-t polynomial evaluations in the field, and a single hash operation. Regarding communication, our DKG protocol entails each party privately sending n field elements to other participants, along with broadcasting approximately $2n$ images generated through hash functions (i.e., the commitments) and t field elements (i.e., coefficients of $r(X)$). Conversely, in the Pedersen DKG protocol, each party privately sends $2n$ field elements to the other participants and then broadcast $2t$ group elements. We refer Table 4 for a summarized comparison of the asymptotic costs in both DKG protocols.

When evaluating the efficiency of threshold signatures, as outlined in our protocol (described in Fig. 5), in addition to executing our DKG protocol with associated costs summarized in Table 4, each participants is required to perform n evaluations of degree-t polynomials in the field (for the verification of partial openings) and broadcast t field elements (representing the coefficients of the polynomial $r^{(i)}(X)$). In the GJKR [21] signature, alongside the Pedersen DKG protocol with its cost breakdown provided in Table 4, each party must conduct $2n$ group exponentiations and broadcast a single field element (for the opening).

In line with the efficiency trends observed in the VSS and DKG protocols, we anticipate that our threshold signature scheme can have significantly improved performance compared to the construction presented by Gennaro et al. [21].

Table 4. Asymptotic costs in the GJKR [21] variant of Pedersen DKG [28] and our proposed scheme. DL: Discrete Logarithm, BC: Broadcast, n: Number of parties, $t \approx n/2$: Threshold value, $E_{\mathbb{G}}$: Exponentiation in group $\mathbb{G}$, $\mathcal{PE}$: degree-t Polynomial Evaluation, $\mathcal{H}$: Hashing, $|\mathbb{G}|$: $\mathbb{G}$ element size, $|\mathbb{Z}_q|$: $\mathbb{Z}_q$ element size, $|\mathcal{H}|$: $\mathcal{H}$ image size.

DKG Scheme	Assumption	Parties' Computation	Communication
GJKR [21] (Pedersen [28])	DL-based	$(2nt+n)\ E_{\mathbb{G}} + 2n\ \mathcal{PE} + 1\ \mathcal{H}$ ($2nt\ E_{\mathbb{G}}$ is for verify)	Private: $2n\ \lvert\mathbb{Z}_q\rvert$ BC: $2t \approx n\ \lvert\mathbb{G}\rvert$
This work Sec. 4.1	DL & Hash-based	$2n\ E_{\mathbb{G}} + 3n\ \mathcal{PE} + 5n\ \mathcal{H}$ ($2n\ E_{\mathbb{G}} + n\ \mathcal{PE} + 3n\ \mathcal{H}$ is for verify)	Private: $1n\ \lvert\mathbb{Z}_q\rvert$ BC: $2n\ \lvert\mathcal{H}\rvert + t\ \lvert\mathbb{Z}_q\rvert$

5 More Efficient Threshold Protocols from Isogenies

Our VSS construction from Section 3 can be seamlessly integrated into current isogeny-based DKG protocols and threshold signatures present in the literature [1, 2, 7, 12]. The resulting protocols outperform the current state-of-the-art in terms of communication and/or computational cost.

In the interest of clarity and conciseness, we defer a brief introduction to isogenies and of the state-of-the-art protocols to App. B. Here, we only present the modifications to these protocols to achieve the better performance results.

5.1 More Efficient DKG Protocols for CSIDH

In this section, we revisit the DKG protocols recently proposed in [2] for CSIDH-based primitives, which can be considered variants of CSI-RAShi [1,7] and Structured CSI-RAShi [1]. We only focus on the structured case (i.e. where the target public key has the structure $\{E_i = [c_i x_0]E_0\}_{i=1}^k$ for public integers $\{c_i\}_{i=1}^k$), as it is generally more efficient and a single public key is a special case of this (for $k = 1$). The extended (non-structured) case (i.e. public keys of the type $\{E_i = [x_i]E_0\}_{i=1}^k$) can be inferred from the latter.

Structured CSI-RAShi++ DKG Protocol. Both DKG protocols proposed in [2] are new variants of the protocols in [1, 7] with lower computational cost which in terms of isogeny computations. This is achieved at the cost of higher communication complexity, a reduced number of corrupted parties to $n/3$, and an interactive share verification in the final DKG protocols.

We revisit these protocols in Fig. 10 of App. B.2. They consist of two stages: a VSS step and a computationally secure public key computation step. During the VSS step, the parties engage in the BGW VSS scheme [4] and share a secret x_0 (i.e., the secret key) among themselves. Then, in the (public key) computation step, they use their shares obtained from the first step and compute the target public key $\{E_i = [c_i x_0]E_0\}_{i=1}^k$ in a round-robin fashion.

In this section, we show that by integrating our new NI-VSS scheme into the VSS step of their DKG protocols, we can resolve all of the drawbacks mentioned above at the same time. Our protocols achieve lower communication, allow $n/2$ corrupted parties and are non-interactively verifiable, while achieving the same computational complexity as the fastest protocols from [2].

In Fig. 6, we present a new variant of the structured DKG protocol from [2, Section 4], by replacing the BGW VSS scheme used in their protocol with our NI-VSS scheme. This replacement results in a reduction of IT security in the VSS step to computational security, but overall, as in the original case, the resulting DKG protocol achieves quantum computational security.

We discuss security below. For formal definitions of the security properties of DKGs, we refer to [7, 21].

Theorem 6 (Structured CSI-RAShi++ DKG Protocol). *If the VSS scheme given in Fig. 2 is a secure verifiable secret sharing scheme in the QROM, then the DKG protocol of Fig. 6 is secure in the QROM. That is correct, robust, and satisfies the secrecy property.*

Verifiable Secret Sharing Step: This is done using the NI-VSS scheme presented in Fig. 2 in a standard distributed manner. Namely, each party P_i one time plays the role of the dealer in Fig. 2, samples $f^{(i)}(X)$, and then in a verifiable manner shares $f^{(i)}(0)$ with other parties. In the end, all the shareholders get a share of the joint secret key x_0, where implicitly is defined as $x_0 = \sum_{i \in Q} f^{(i)}(0)$ for a qualified set Q. Each party P_j obtains its share of x_0 as $x_j = \sum_{i \in Q} f^{(i)}(j)$.

SPK Computation Step: This is done as in the structured public key computation step of the DKG protocol presented in [2, Section 4], which is reviewed in Fig. 10. At the end, the parties return the structured public key $\{E_i = [c_i x_0]E_0\}_{i=1}^{k}$.

Fig. 6. Structured CSI-RAShi++: an efficient DKG protocol for a structured public key $\{E_i = [c_i x_0]E_0\}_{i=1}^{k}$.

Proof. The proof is almost identical to the proof of [2, Theorem 4.1], except that in this case we will rely on the security of the new NI-VSS, proven in Theorem 2, rather than the security of the BGW VSS scheme [4] which is employed in the secret sharing step of their DKG protocol. □

Efficiency of the Revised DKG Protocols. In Tables 5 and 6, we summarize the computational and communication costs of our proposed DKG protocols, CSI-RASHI++ and Structured CSI-RAShi++ (both from Fig. 6, the former for the choice $k = 1$), and compare them with current DKG protocols in the CSIDH setting. To have a fair comparison we express the computational cost as the sequential runtime of the protocol steps, i.e. the total runtime from start to finish, including when some of the parties are idle. We quantify the communication cost as the amount of outgoing communication per party. Our cost analysis methodology builds on that of [2] with some optimizations from [1].

Table 5. Sequential computational costs (including idle time) of the different DKGs from [2] and from this work, in terms of polynomial evaluations, isogeny computations and calls to the commitment scheme and random oracle. For compactness, we assume $\gcd(k, n) = \min\{k, n\}$ and do not explicitly write down the gains through the twist trick. See [2] for more details.

	Polynomial Eval.	Isogenies	Commitments	RO queries
Basic DKG [2]	$2(n-1)^2 + n\lambda(n+2)$	$2n\lambda + n$	$2n(n+3)$	$2n$
Extended DKG [1,2]	$2(n-1)^2 k + n\lambda(n\lceil \frac{k}{n} \rceil + k)$	$n(n\lambda+1)\lfloor \frac{k}{n} \rfloor$	$2n((n-1)\lceil \frac{k}{n} \rceil + 2k)$	nk
Structured DKG [2]	$2(n-1)^2 + n\lambda(2n+1)$	$n(n\lambda+1)\lfloor \frac{k}{n} \rfloor$	$2n(3n-1)$	n^2
Our Basic DKG	$(3n-1) + n\lambda(n+2)$	$2n\lambda + n$	$2n(n+5) - 2$	$3n$
Our Extended DKG	$(3n-1)k + n\lambda(n\lceil \frac{k}{n} \rceil + k)$	$n(n\lambda+1)\lfloor \frac{k}{n} \rfloor$	$2n((n-2)\lceil \frac{k}{n} \rceil + +4k) - 2k$	$2nk$
Our Structured DKG	$(3n-1) + 2n^2\lambda$	$n(n\lambda+1)\lfloor \frac{k}{n} \rfloor$	$2n(3n+1) - 2$	$n(n+1)$

Table 6. Communication costs of different DKGs from [2] and this work, in terms of elements in $\mathbb{Z}_N$ and $\mathcal{E}$, and the number of commitments and proof pieces (i.e. elements of size 2λ). The cost represents the outgoing cost per party. The cost of the basic DKG follows by setting $k = 1$.

	Element of Z_N	Element of $\mathcal{E}$	Commitment/Proof Piece
Extended DKG [1, 2]	$2k(n-1)(n+t-1)$ $+kn\lambda(t+1)$	nk	$nk(3n+2)$
Structured DKG [2]	$2(n-1)(n+t-1)$ $+n\lambda(t+1)$	nk	$n(3n+2)$
Our Extended DKG	$k(n\lambda(t+1)+n+t)$	nk	$k(n(3n+5)-1)$
Our Structured DKG	$n\lambda(t+1)+n+t$	nk	$n(3n+5)-1$

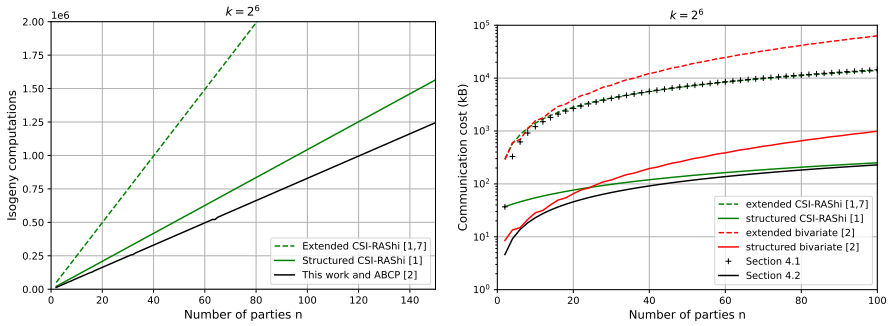

Fig. 7. Computational and communication costs of the DKG protocols [1, 2, 7] for the CSIDH-512 parameter set, shown as a function of the number of parties for $k = 2^6$.

In Fig. 7, we further plot the computational and communication costs of our protocols and compare them to the literature. We note that the number of isogeny computations coincides exactly with [2], currently the fastest in the literature. In terms of communication, both the extended and structured versions of our protocols outperform their counterparts from the literature. For asymptotically large n, the communication cost of our protocols tend towards the communication cost of CSI-RAShi, as the commmunication cost of underlying NI-TZK proof schemes starts to dominate in these regions.

5.2 Threshold, Efficient, and Robust CSI-SharK

Next, we revisit the CSI-FiSh-based threshold signing protocol of Campos and Muth [12], and construct *ThreshER SharK*, which is a **Thresh**old, **E**fficient and **R**obust signature scheme based on CSI-**SharK** [1]. Campos-Muth threshold signature [12] is based on the basic version of CSI-FiSh [8], with its public key and the ephemeral keys being sampled by the CSI-RAShi DKG protocol [7]. The basic version of CSI-FiSh is based on an ID scheme that has a binary challenge space, leading to long signing and verification times. By using larger public keys

KeyGen: Given an integer k, as the design parameter in CSI-SharK signature, the parties first agree on a public exceptional set $\Xi_k = \{c_0 = 0, c_1 = 1, c_2, \ldots, c_{k-1}\}$. Then, they engage in the $\mathsf{DKG}^{\text{Structured CSI-RAShi++}}$ protocol (refer to Fig. 6) to sample the public key $\mathsf{pk} := (E_0, \ldots, E_{k-1})$, where $E_i = [c_i x]E_0$. At the end, we have a qualified set, which w.l.o.g. we assume to be $Q_0 := \{P_1, P_2, \ldots, P_n\}$.

Sign$(m, \langle x_i \rangle)$**:** To generate a signature on m, the parties in Q_0 act as follows.

1. For $l = 1, \ldots, t_k$, parties run $(b_l, F_l, B_l) \leftarrow \mathsf{DKG}^{\text{CSI-RAShi++}}(E_0)$. Note that at each invocation, malicious parties might be disqualified. We assume to end every step with a set $Q_l \subseteq Q_{l-1}$. Only the parties in Q_{t_k} continue the protocol.
2. The parties compute the challenge $d_1, \ldots, d_{t_k} \leftarrow H(F_1, \ldots, F_{t_k} || m)$.
3. For $l = 1, \ldots, t_k$, the parties in Q_{t_k} behave as follows.
   (a) each party P_i computes $r_l^{(i)}(X) = b_l^{(i)}(X) - d_l s^{(i)}(X)$.
   (b) using their secret values shared during the NI-VSS protocol, namely $s^{(i)}$ and $b_l^{(i)}$, each other party P_j verifies
   $$r_l^{(i)}(j) \stackrel{?}{=} b_l^{(i)}(j) - d_l s^{(i)}(j)$$
   (c) Whenever one of these checks fails, P_j broadcasts a complaint against P_i. When a player P_i has $t+1$ or more complaints against them, they are disqualified. The remaining players can then construct $r_l^{(i)}(0)$ by reconstructing both $b_l^{(i)}(0)$ and $s^{(i)}(0)$ using the information from the DKGs. This is always possible when there are at least $t+1$ honest parties.
   (d) For each party P_i in $Q_l \backslash Q_{t_k}$, reconstruct $r_l^{(i)}(0)$ in the same way.
   (e) For each party P_i in $Q_0 \backslash Q_l$, set and reconstruct $r_l^{(i)}(0) = s^{(i)}(0)$.
   (f) Using $\{r_l^{(i)}(0)\}_{i=1,\ldots,n}$, parties build the responses $r_l = \sum_{i \in Q} r_l^{(i)}(0)$.
4. Finally, parties output the signature (r, d), where $r = (r_1, \ldots, r_{t_k})$.

Verify$((r, d_1, \ldots, d_{t_k}), m, \mathsf{pk})$**:** To verify a signature $(r, d_1, \ldots, d_{t_k})$ on m using the public key $\mathsf{pk} = (E_1, \ldots, E_{k-1})$, the verifier proceeds as follows.

1. For $l = 1, \ldots, t_k$, compute $F_l = [r_l]E_{d_l}$, where
2. Compute $d'_1, \ldots, d'_{t_k} \leftarrow H(F_1, \ldots, F_{t_k} || m)$
3. If $d_1 = d'_1 \wedge \cdots \wedge d_{t_k} = d'_{t_k}$, return valid, otherwise invalid.

Fig. 8. ThreshER SharK: a Threshold, Efficient, and Robust signature scheme based on CSI-SharK.

(e.g. extended or structured ones) of k elements, less repetitions are needed and the signatures become faster and smaller.

Similarly, one can extend the robust threshold signing protocol of Campos and Muth to work with the extended version of CSI-FiSh [8] or its structured counterpart CSI-SharK [1]. Based on the comparisons presented in Tables 5-6 and Fig. 7, we know that the latter has faster DKG protocols to generate the public key. Considering this, we propose two modifications to enhance the efficiency of Campos and Muth's robust threshold signature scheme [12]. First, we adapt their threshold signature scheme to work with CSI-SharK [1]. Consequently, the parties can use the Structured CSI-RAShi++ DKG protocol to sample the keys and need to store only a single secret key. As the second modification, we employ the (non-structured) extended variant of our pro-

posed CSI-RAShi++ DKG protocol to sample the ephemeral keys for the revised threshold signature scheme. Fig. 8 describes the algorithms of the resulting robust threshold signature, which is called ThreshER SharK. In the figure, $\mathcal{H} : \{0,1\}^* \to (\Xi_k)^{t_k}$ is a random oracle which returns t_k elements from an exceptional set $\Xi_k = \{c_0 = 0, c_1 = 1, c_2, \ldots, c_{k-1}\}$ of size k. ThreshER SharK uses the new NI-VSS scheme in both the key generation and singing protocols.

Efficiency. ThreshER SharK, utilizing an SPK, benefits from the ability to sample keys more efficiently using Structured CSI-RAShi++. While it is possible to extend Campos and Muth's robust threshold signature scheme [12] to accommodate the extended version of CSI-FiSh and gain efficiency through our proposed DKG protocols, it should be noted that the result would be less efficient than ThreshER SharK (We refer to Tables 5-6 for a detailed comparison).

Security. We discuss security of our scheme in App. B.3.

6 Conclusion

In this paper, we presented a general construction for building a NI-VSS scheme using a ZK proof scheme over secret shared data, as formally defined by Boneh et al. [10]. Leveraging this construction, we proposed a practical post-quantum secure NI-VSS scheme based on Shamir secret sharing.

The proposed NI-VSS scheme can be viewed as a modification of the variant of the Pedersen VSS scheme used in the GJKR DKG protocol [21], where we replace the Pedersen commitment with a hash-based commitment scheme. The later modification pushes the protocol to the designated verifier setting, requires a ZK proof over secret shared data, but enables the attainment of Post-Quantum (PQ) security and notably improved efficiency. Consequently, our NI-VSS scheme presents a more efficient and PQ-secure alternative to the Pedersen [28] (or Feldman [19]) VSS scheme in scenarios where public verifiability is not necessary. This holds true for various (post-quantum secure) threshold protocols such as DKG schemes and threshold signatures. To assess the performance of the new NI-VSS scheme alongside the well-established Pedersen scheme, we conducted a prototype implementation, yielding promising results. Compared to the Pedersen scheme, while incurring a 2.5× higher broadcast cost for the dealer, our scheme demonstrates significantly faster sharing times, ranging from $32.5 - 3.25\times$ faster for different values of (n, t), spanning from $(32, 15)$ to $(2048, 1023)$. Moreover, verification times are substantially reduced, with the new scheme requiring $85 - 479\times$ less time. Using our scheme, a dealer can share a secret with 2048 parties in approximately 2 sec, while parties can verify their shares in less than 5 msec.

A key advantage of new PQ-secure NI-VSS scheme, when compared to the IT-secure BGW VSS scheme [4], is its reduced communication overhead and improved robustness. Specifically, our scheme requires half of the shareholders to be honest, as opposed to two-thirds in the BGW VSS scheme.

Leveraging the new NI-VSS scheme we revisited and improved various classic and PQ-secure DKG and threshold signing protocols [1,2,7,12,21]. Through

our revisions, we have not only improved their performance but also relaxed the requirements on the number of honest parties in some cases. We have introduced a new variants of the Pedersen DKG [28] combined with the GJKR threshold signature scheme [21] that can outperform the original versions in terms of computational cost. One notable factor contributing to these improvements is the practical advantage of using a hash function for commitment, which often outperforms the Pedersen commitment. Our results show that, in practice, DKG and threshold signing protocols with designated verifiers suffice for constructing a threshold signature scheme with public-verifier. Our revisions also have led to the development of two DKG protocols and a threshold signature scheme based on isogenies that surpass the state-of-the-art constructions [2, 12]. Our isogeny-based threshold signature scheme builds upon the CSI-SharK signature [1], but it can also be adapted to work with the CSI-FiSh [8], although with lower efficiency in the key generation phase.

The remarkable efficiency and simplicity of the NI-VSS scheme render it a valuable tool for various classic and PQ-secure threshold protocols, extending beyond those revisited in this paper. Future research can explore the integration of the new NI-VSS scheme and the revised protocols into other settings and threshold protocols, while evaluating their impact on overall efficiency.

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A Security of the Schnorr-based Threshold Signature

The proof is analogous to that of Gennaro et al. [21]. Let $\mathcal{A}$ be an adversary that breaks the unforgeability of the threshold signing protocol in Fig. 5. Then we construct a simulator Sim that uses $\mathcal{A}$ as a subroutine to break the DL problem.

Specifically, the challenger for the DL problem gives h_T to Sim which needs to find x_T such that $h_T = g^{x_T}$. The strategy is to embed h_T into the public key computation of the protocol in Fig. 5. For simplicity[9], assume that the set of malicious parties is $A = \{P_1, \ldots, P_t\}$ while the set of honest parties is $H = \{P_{t+1}, \ldots, P_n\}$.

Sim simulates the parties in H correctly except one, say P_n. In other words, for $P_{t+1}, \ldots, P_{n-1}$ it follows the DKG protocol as an honest party would, therefore producing a partial public key h_i. While for party P_n, it simulates the NI-TZK in such a way that h_T is the partial public key of P_n.

To sign a message m, Sim samples $r_n, c \leftarrow \mathbb{Z}_p$ uniformly at random, and sets $z_n := g^{r_n} h_T^{-c}$. Then Sim simulates the DKG protocol for computing the ephemeral key z the same way as it did for the public key, by forcing the contribution of P_n to be z_n. It finally programs the RO so that $c = H(z\|m)$. The simulation of the ephemeral key succeeds except when the query (z, m) is queried to the RO before Sim gets the value z. One can prove that this only happens with negligible probability. After that, Sim simply follows the protocol normally, using the shares r_i that it computed honestly and r_n to compute the response $r = \sum_i r_i$, therefore simulating the transcripts of the protocol.

If the adversary is able to forge a signature, say (c, r) on a message m with non-negligible probability ε, then Sim re-winds $\mathcal{A}$ to the point where it asked the query $(z\|m)$ to H, where $z = g^r y^{-c}$. From that point, the simulator keeps simulating the protocol with fresh randomness and eventually will get a new forgery (c', r') on m with $(c' \neq c)$. The probability that this happens is non-negligible in ε by the forking lemma. This means that, with non-negligible probability, Sim gets two pairs $(c, r), (c', r')$ such that

$$z = g^r h^{-c} = g^{r'} h^{-c'},$$

or, by expanding each factor,

$$g^r \left(\prod_{i=1}^{n-1} (g^{x_i})^{-c} \right) \cdot h_T^{-c} = g^{r'} \left(\prod_{i=1}^{n-1} (g^{x_i})^{-c'} \right) \cdot h_T^{-c'}$$

[9] This is without loss of generality in that the protocol is symmetric for all parties

from which Sim can compute the discrete logarithm of h_T as $x_T = \frac{r-r'}{c'-c} - \sum_{i=1}^{n-1} x_i$, given that it has all the shares x_i for $i \neq n$.

B More Efficient Threshold Protocols from Isogenies

In this section, we first start with a brief introduction to isogeny-based cryptography and then present the current CSIDH-based DKG protocols and threshold signatures scattered throughout the literature [1,2,7,12].

B.1 Isogeny-based Cryptography

Isogenies are rational maps between elliptic curves that are also homomorphisms with respect to the natural group structure on these curves. Our investigation is limited to the set $\mathcal{E}$ of supersingular elliptic curves over prime fields $\mathbb{F}_p$ and separable $\mathbb{F}_p$-rational isogenies defined between them (the so-called CSIDH setting). Isogenies from an elliptic curve to itself are called endomorphisms. Under the addition and composition operations, the endomorphisms of elliptic curves form a ring. The subring of $\mathbb{F}_p$-rational endomorphism rings of curves in $\mathcal{E}$ is always isomorphic to an order $\mathcal{O}$ in the quadratic imaginary field $\mathbb{Q}(\sqrt{-p})$. Separable isogenies are uniquely defined by their kernel, which can be identified with the kernels of ideal classes in the ideal-class group $\mathrm{cl}(\mathcal{O})$. As a result, we can see the class group as acting on the set $\mathcal{E}$ via a free and transitive group action.

To ensure efficient computation of isogenies, the prime p is usually chosen such that $p - 1 = 4\prod_i \ell_i$, where the ℓ_i are small prime factors. The factor 4 ensures that $p \equiv 3 \bmod 4$ and that the special elliptic curve $E_0 : y^2 = x^3 + x$ is supersingular. Throughout this work, we assume that the class group $\mathrm{cl}(\mathcal{O})$ is known, enabling the transformation of arbitrary ideals into efficiently computable isogeny chains of degrees l_i using the relation lattice. We note that this is not a trivial assumption as current class group computations in reach fall short of realistic security levels [8,11,29] or lead to very slow protocols [17]. We point out, however, that there are polynomial-time quantum algorithms to this end [25]. We refer to [6,8,13,39] for more details on the explicit computations of isogenies. For a more thorough introduction to isogenies and isogeny-based cryptography, we recommend [13,16,35].

Finally, we note that class groups are generally of composite order. By working in cyclic subgroups of $\mathrm{cl}(\mathcal{O})$ with generator $\mathfrak{g}$ and order $N \mid \#\mathrm{cl}(\mathcal{O})$, we can redefine the group action as $[\,] : \mathbb{Z}_N \times \mathcal{E} \to \mathcal{E}$, where ideals of the form $\mathfrak{g}^a$ for $a \in \mathbb{Z}_N$ can be reduced modulo the relation lattice and efficiently computed. To work in a subgroup $\mathbb{Z}_{N'} \subset \mathbb{Z}_N$, we can simply use the generator $\mathfrak{g}^{N/N'}$. For the rest of this work, we always assume the choice of the subgroup $\mathbb{Z}_N$ to be such that $\{1, \ldots, n\}$ defines an exceptional set modulo N, i.e. that n is smaller than the smallest divisor of N.

B.2 (Structured) PVPs and DKG Protocols

(Structured) Piecewise Verifiable Proofs. We provide a brief overview of Piecewise Verifiable Proofs (PVPs) [7], as well as two available constructions that are utilized in the various threshold schemes from [1, 2, 7, 12]. PVPs are particular ZK proofs over secret shared data [10] that similarly consist of a collection of distributed relations $R_0, \ldots, R_n$, with the same witness space, where each statement can be verified independently. The goal of a PVP is to prove the existence of a witness w that satisfies $(x_i, w) \in R_i$ for every $i \in \{0, \ldots, n\}$, given a list of statements $x_0, \ldots, x_n$. The proof itself takes the form of $(\tilde{\pi}, \pi_i)_{i\in\{0,\ldots,n\}}$, where $(\tilde{\pi}, \pi_0)$ enables verification of x_0 in relation to R_0 (also known as the main or central proof), and π_i for $i \in \{1, \ldots, n\}$ enables verification of x_i in relation to R_i. Roughly speaking, PVPs [7] can be considered as a special case of NI-TZK proofs over shared data [10], when the proof piece $(\tilde{\pi}, \pi_0)$ associated to R_0 is named the central proof, and the $(\tilde{\pi}, \pi_i)_{i\in\{0,\ldots,n\}}$ are the proof pieces that are only relevant for $\{R_i\}_{i\in\{0,\ldots,n\}}$. In [2, 7], the authors have presented PVP schemes for the following list of n-distributed relations $R = (R_0, \ldots, R_n)$,

$$R_0 = \{((\Xi_k, F_1, F_1', \ldots, F_k, F_k'), f(x)) \mid (F_l' = [c_l f(0)]F_l)_{l=1}^{k}\ \},$$
$$\forall i = 1, \ldots, n : R_i = \{(x_i, f(x))|f(i) = x_i\} \tag{5}$$

where the statement of R_0 consists of a public exceptional[10] set $\Xi_k = \{c_1 = 1, c_2, \ldots, c_k\}$ and a set of curves $(F_1, F_1', \ldots, F_k, F_k') \in \mathcal{E}^{2k}$, and the statements for the relations $\{R_i\}_{i=1,\cdots,n}$ are elements of $\mathbb{Z}_N$. The witnesses for both PVP schemes is a secret polynomial $f(X) \in \mathbb{Z}_N[X]_t$, which is a polynomial in the variable X with coefficients defined over $\mathbb{Z}_N$ and have a maximum degree of t. In fact, the PVP scheme proposed in [7], is a special case of the PVP scheme proposed in [2], when $k = 1$, i.e., $\Xi_1 = \{c_1 = 1\}$ [11].

Fig. 9 describes the prover and verification algorithms of the PVP schemes proposed in [2, 7], where $\mathcal{H} : \{0,1\}^* \to \{0,1\}^\lambda$ is a random oracle and $\mathcal{C} : \{0,1\}^* \times \{0,1\}^\lambda \to \{0,1\}^{2\lambda}$ is a commitment scheme that is collapsing [37, Definition 12] and quantum computationally hiding [7, Definition 2]. In [2, 7], the authors show that their PVP schemes are correct, sound (against the prover and t malicious verifiers) and ZK (i.e., TZK, ZK against t malicious verifiers), for the relation in equation (5) Their PVP schemes [2, 7] have a binary/ternary challenge space, and allow a prover to convince a set of verifiers that she knows the secret element $f(0) \in \mathbb{Z}_N$ that connects a pair or a set of elliptic curves with correct factors through the group action $[f(0)]$, and where each verifier has a share of $f(0)$.

Robust DKG Protocols for CSIDH. DKG protocols allow a group of parties to generate a secret and public key pair in a fully distributed manner. The

[10] We can use superexceptional sets in the case, where all $F_1 = \cdots = F_k = E_0$, where $E_0 : y^2 = x^3 + x$, see [3].

[11] Note that a PVP scheme is a NI-TZK proof for its underlying n-distributed relation, and actually in both PVP schemes [2, 7], the authors implicitly prove the TZK and soundness against the prover and t malicious verifiers, but do not call them as such.

Prover: Given, a witness polynomial $f(X) \in \mathbb{Z}_N[X]_t$ and a statement $x = ((\Xi_k, F_1, F'_1, \ldots, F_k, F'_k), x_1, \cdots, x_n))$, outputs a non-interactive piecewise proof π of the relations in equation (5).

1. Parse $\Xi_k = \{c_1, c_2, \ldots, c_k\}$.
2. For $j = 1, \ldots, \lambda$: sample $b_j \leftarrow \mathbb{Z}_N[X]_t$ and compute $\hat{F}_j^1 = [c_1 b_j(0)]F_1 \ , \ldots, \ \hat{F}_j^k = [c_k b_j(0)]F_k$
3. Sample $y_0, y'_0 \leftarrow \{0,1\}^\lambda$ uniformly at random;
4. Set $\mathsf{C}_0 = \mathcal{C}(\hat{F}_1^1, \cdots, \hat{F}_1^k \parallel \cdots \parallel \hat{F}_\lambda^1, \ldots, \hat{F}_\lambda^k, y_0)$,
5. Set $\mathsf{C}'_0 = \mathcal{C}(F_1, F'_1 \parallel \cdots \parallel F_k, F'_k, y'_0)$.
6. For $i = 1, \ldots, n$: sample $y_i, y'_i \leftarrow \{0,1\}^\lambda$ and set $\mathsf{C}_i = \mathcal{C}(b_1(i) \parallel \cdots \parallel b_\lambda(i), y_i)$ and $\mathsf{C}'_i = \mathcal{C}(x_i, y'_i)$;
7. $\mathbf{d} = d_1 \ldots d_\lambda = \mathcal{H}(\mathsf{C}, \mathsf{C}')$, where $\mathsf{C} = (\mathsf{C}_0, \ldots, \mathsf{C}_n), \mathsf{C}' = (\mathsf{C}'_0, \ldots, \mathsf{C}'_n)$;
8. For $j = 1, \ldots, \lambda$: compute $r_j(x) = b_j(x) - d_j f(x) \mod N$
9. Return $\tilde{\pi} = (\mathsf{C}, \mathsf{C}', \mathbf{r})$ and $\{\pi_i = (y_i, y'_i)\}_{i=0}^n$, where $\mathbf{r} = (r_1, \ldots, r_\lambda)$.

Verification: Given, an index $i \in \{0, \ldots, n\}$, a statement piece x_i of the form $x_0 = (\Xi_k, F_1, F'_1, \cdots, F_k, F'_k)$ if $i = 0$, or $x_i \in \mathbb{Z}_N$ if $i \neq 0$, as well as a proof piece $(\widetilde{\pi}, \pi_i) = ((\mathsf{C}, \mathsf{C}', \mathbf{r}), (y_i, y'_i))$, outputs `true` or `false`.

1. If $\mathsf{C}'_i \neq \mathcal{C}(x_i, y'_i)$, then return `false`
2. $d_1 \ldots d_\lambda = \mathcal{H}(\mathsf{C}, \mathsf{C}')$;
3. If $i == 0$:
   - For $j = 1, \ldots, \lambda$:
     - If $d_j == 0$: $\widetilde{F}_j^1 \leftarrow [c_1 r_j(0)]F_1, \ \ldots \ , \widetilde{F}_j^k \leftarrow [c_k r_j(0)]F_k$,
     - else $\widetilde{F}_j^1 \leftarrow [c_1 r_j(0)]F'_1, \ \ldots \ , \widetilde{F}_j^k \leftarrow [c_k r_j(0)]F'_k$
   - return $\mathsf{C}_0 == \mathcal{C}(\widetilde{F}_1^1, \cdots, \widetilde{F}_1^k \parallel \cdots \parallel \widetilde{F}_\lambda^1, \cdots, \widetilde{F}_\lambda^k, y_0)$
4. Else, return $\mathsf{C}_i == \mathcal{C}(r_1(i) + d_1 x_i \parallel \cdots \parallel r_\lambda(i) + d_\lambda x_i, y_i)$

Fig. 9. The prover and verification algorithms of the PVP schemes proposed in [7] (when $k = 1$, i.e., $\Xi_1 = \{c_1 = 1\}$) and [2].

property of robustness implies that this key pair generation always succeeds, even in the presence of malicious parties that can behave arbitrarily. In [7], Beullens, Disson, Pedersen, and Vercauteren constructed the first robust DKG protocol for CSIDH, called CSI-RAShi, that allows a set of parties to sample public key $F_1 = [x_0]E_0$ in a distributed manner, with x_0 being their Shamir secret shared value. CSI-RAShi works in the majority honest setting and consists of two steps. In the first step, each party P_i samples a polynomial $f_i(X) \in \mathbb{Z}_N[X]_t$, where $t \leq \lfloor \frac{n-1}{2} \rfloor$, and publishes $F_i = [f_i(0)]F_0$, while sharing the values $f_i(j)$ among the other parties $\{P_j\}_{j \in [n]}$. To ensure the correctness of the shared values, parties publish a PVP (i.e., a NI-TZK proof) for the relations $R_0, \ldots, R_n$ as described earlier (in Fig. 9 for $k = 1$, i.e., $\Xi_1 = \{c_1 = 1\}$). Then, each other party P_j verifies "their" relation R_j as well as the main relation R_0. In an honest-majority setting, if all parties agree that their shares are correct, they can be certain that each one of them possesses a share of a unique polynomial $f_i(X)$ of degree t, whose evaluation in 0 is in committed in $F_i = [f_i(0)]F_0$. In the next step, parties use their committed values and in a round-robin manner compute the final public key $[x_0]E_0$ and attach a ZK proof for their correct action (i.e., updating the PK correctly).

In follow-up work [1], Atapoor, Baghery, Cozzo, and Pedersen presented a new variant of the CSI-RAShi DKG protocol (so-called Structured CSI-RAShi), that allows a set of parties to sample a structured public key (SPK), i.e. a public key of the type $\{[c_i x_0]E_0\}_{i=1}^{k}$, where $\Xi_k := \{c_1 = 1, c_2, \cdots, c_k\}$ is an exceptional set, in a distributed manner. Structured CSI-RAShi also works in the majority-honest setting and allows a set of parties to sample an SPK at least $3\times$ faster than the case that one uses the extended and optimized version of CSI-RAShi [1]. Similar to CSI-RAShi, the structured CSI-RAShi also consists of two phases. Its first phase is the same as in the basic CSI-RAShi, but in the second phase, parties use multiples of their committed values $f_i(0)$ and in a round-robin manner compute the final SPK and attach a ZK proof for their correct action (i.e., updating the SPK correctly). Using multiples of the same secret allows to summarize multiple related statements into a single ZK proof.

In a different work [2], the same authors presented two new DKG protocols for CSIDH-based primitives, but using the BGW VSS scheme [4]. The latter uses bivariate polynomials and is very efficient and achieves information-theoretical security, but it needs at least $2/3$ of the parties to be honest and also requires interaction between the verifiers (shareholders) to check the validity of the shares. Then, in the second phase of their DKG protocols, parties use their secret shared value from the previous phase, namely $f_i(0)$, and engage in a round-robin public key computation protocol. In order to prove correct action, parties can now use PVPs instead of ZK proofs. The authors propose two protocols, one using (extended) public keys, and one using SPKs. For the latter, the structured PVPs from Fig. 9 are used. Fig. 10 summarizes both their DKG protocols and highlights the differences. Note that the first phases (the VSS) are identical.

B.3 On the Security of ThreshER SharK

Here we give an intuitive proof of the security of the protocol in Fig. 8. The proof is a reduction-based argument, which means that we show that given an adversary that breaks the sEU-CMA security of the signing protocol in Fig. 8, then we can construct an efficient algorithm $\mathcal{S}$ that solves a hard problem. The problem will be the one behind the CSI-SharK signature scheme, which is the Vectorization Problem with Auxiliary Inputs (VPwAI) [3]. The proof is analogous to the DL-based one given in [21, Theorem 2].

Assume $\mathcal{A}$ is an adversary controlling up to t parties in the protocol in Fig. 8 and that breaks the unforgeability property of the threshold signature. The simulator $\mathcal{S}$ receives a set of k structured supersingular elliptic curves along with the exceptional set $\Xi_k = \{c_0 = 0, c_1 = 1, \cdots, c_{k-1}\}$,

$$\left\{\Xi_k, E_0, E_1^* = [c_1 x^*]E_0, \cdots, E_{k-1}^* = [c_{k-1}x^*]E_0\right\}$$

from the challenger. He needs to find the secret isogeny x^*. The simulator embeds the elliptic curves he got from the challenger into the protocol for computing the public key. Specifically, he simulates the DKG protocol, so that the resulting public key is $E_0, E_1^* = [c_1 x^*]E_0, \cdots, E_{k-1}^* = [c_{k-1}x^*]E_0$. To do this, like in the

Verifiable Secret Sharing (BGW VSS) [4]:

1. For $i = 1, \dots, n$, player P_i
   (a) samples $q^{(i)}(Z) \leftarrow \mathbb{Z}_N[Z]_t$ and sets $x^{(i)} = q^{(i)}(0)$,
   (b) samples $S^{(i)}(X, Y) \leftarrow \mathbb{Z}_N[X, Y]_t$ with $S^{(i)}(0, Z) = q^{(i)}(Z)$,
   (c) for $j = 1, \dots, n$, defines $f_j^{(i)}(X) = S^{(i)}(X, j)$ and $g_j^{(i)}(Y) = S^{(i)}(j, Y)$ and sends $\{f_j^{(i)}(X), g_j^{(i)}(Y)\}$ privately to party P_j.
2. For $k = 1, \dots, n$, each pair of players P_i, P_j checks that $f_i^{(k)}(j) = g_j^{(k)}(i)$ and $g_i^{(k)}(j) = f_j^{(k)}(i)$. Whenever one of these checks fails, the concerned player runs the conflict resolution procedure of BGW VSS scheme (reviewed in [2, Section 3.1]. In case the procedure outputs $\perp$, the dealer P_k of the concerned polynomials is disqualified, otherwise the protocol continues normally.
3. In the end, all the honest players agree on the same set of qualified players $Q \subseteq \{1, \dots, n\}$, and the shared secret key x is given as the sum of the individual secrets of the qualified players $x = \sum_{i \in Q} x^{(i)}$, while the parties' shares of x can be constructed as $x_j = \sum_{i \in Q} f_j^{(i)}(0)$.

Computing the Single/Structured Public Key:

4. Let $Q = \{1, \dots, n'\}$. Given a superexceptional set $\Xi_k = \{c_1 = 1, c_2, \dots, c_k\}$, a qualified set of parties engage in a round-robin protocol, and party P_i computes
$$F_i^1 \leftarrow [x^{(i)}]F_{i-1}^1, \; F_i^2 \leftarrow [c_2 x^{(i)}]F_{i-1}^2, \; \dots, \; F_i^k \leftarrow [c_k x^{(i)}]F_{i-1}^k,$$
where $F_0^1 = F_0^2 = \cdots = F_0^k = E_0$. At each step, party P_i also uses the Structured PVP scheme given in Fig. 9, and creates and publishes a PVP proof,
$$\pi^{(i)} = (\tilde{\pi}^{(i)}, \pi_1^{(i)}, \dots, \pi_{n'}^{(i)}) \leftarrow \mathbf{Prover}\big(\Xi_k, (F_{i-1}^1, F_i^1, \cdots, F_{i-1}^k, F_i^k); q^{(i)}(Z)\big)$$
which includes a main proof $(\tilde{\pi}^{(i)}, \pi_0^{(i)})$ as well as individual proof pieces $\pi_j^{(i)}$ for each other player P_j. Note that in a single public key generation, they run the PVP scheme in Fig. 9 for $k = 1$, i.e., $\Xi_1 = \{c_1 = 1\}$.
5. Each other player P_j verifies both $\mathbf{Prover}(j, f_j^{(i)}(0), \tilde{\pi}^{(i)}, \pi_j^{(i)})$ and $\mathbf{Verifier}(0, (\Xi_k, (F_{i-1}^1, F_i^1, \cdots, F_{i-1}^k, F_i^k)), \tilde{\pi}^{(i)}, \pi_0^{(i)})$. Whenever a verification of $\pi^{(i)}$ fails, the verifier P_j broadcasts $f_j^{(i)}(X)$. All other parties verify the correctness of $f_j^{(i)}(X)$ as in step 2. If it is correct, since there are at least t honest players, they will be able to reconstruct $q^{(i)}(0)$, compute F_i and proceed with the protocol (and potentially disqualify P_i). Otherwise, if the checks of $f_j^{(i)}(0)$ fail, the complaint can be ignored (or P_j disqualified). In the latter case, the shares of P_j can also be reconstructed by the at least t honest players.
6. At the end of the round-robin, the parties return the *structured* public key
$$F_{n'}^1 = [x_0]E_0, \; F_{n'}^2 = [c_2 x_0]E_0, \; \cdots, F_{n'}^k = [c_k x_0]E_0 \,.$$

Fig. 10. The DKG protocols of Atapoor, Baghery, Cozzo, and Pedersen for a single [2, Section 3] and an SPK generation [2, Section 4]. Both schemes use the BGW VSS [4] for secret sharing. The highlighted terms are particular for single PK computation.

proof of the protocol in Fig. 6, he follows the protocol faithfully for all honest parties except one, say P_{j^*}, for which he sets $E_i^j = E_i^*$ for $i = 1, \cdots, k-1$. This way, the contribution of party P_{j^*} is implicitly set to be $x^* - \sum_{i=1}^{j^*-1} x_i$.

After the key generation, the simulator samples $d = (d^1, \cdots, d^{t_k})$ and $r = (r^{(1)}, \cdots, r^{(t_k)})$ uniformly at random. Then, he computes $F_i = [d^{(i)}]E_{d_i}$ for each $i = 1, \cdots, t_k$ and programs the random oracle so that

$$d = H(F_1 \| \cdots \| F_{t_k} \| m).$$

Then the simulator plays the role of the honest parties in step 1 of Fig. 8 so as to force the ephemeral curves to be $F_{(1)}, \cdots, F_{(t_k)}$, for each one following exactly the same strategy as in the proof of Fig. 6. This succeeds except if the query $(F_1, \cdots, F_{t_k}, m)$ was already asked the RO and the output was not $d_1, \cdots, d_{t_k}$. One can prove that this probability is negligible.

After successfully simulating the ephemeral keys, the simulator can easily simulate the partial responses. For $r_i^{(j)}$ with $i \neq j^*, j = 1, \cdots, t_k$, the simulator computes it by using the shares x_i and $b_i^{(j)}$ it either chose or got from the adversary. For $r_{j^*}^{(j)}$, the simulator simply computes $r_{j^*}^{(j)} = r^{(j)} - \sum_{i \neq j^*} r_i^{(j)}$. This concludes the simulation part.

In order to extract the secret isogeny x^*, the simulator follows the following strategy. After getting a forgery $(d_1, \cdots, d_{t_k}), (r^{(1)}, \cdots, r^{(t_k)})$ on a message m, he rewinds the adversary to the point where he makes the query $(F_1 \| \cdots, F_{t_k} \| m)$ to H. Then he continues to simulate from that point on with fresh randomness. By the forking lemma, with non-negligible probability, the adversary will produce a new forgery $(\tilde{d}_1, \cdots, \tilde{d}_{t_k}), (\tilde{r}^{(1)}, \cdots, \tilde{r}^{(t_k)})$, for the same message m. Then, the simulator computes

$$x^* = \frac{r^{(l)} - \tilde{r}^{(l)}}{\tilde{d}^{(l)} - d^{(l)}} - \sum_{j \neq j^*} x_j,$$

which is a solution for the VPwAI.