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Approximations to the Halting Problem

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The halting set $K_\phi = \{x \mid \phi_x(x) \text{ converges}\}$, for any Gödel numbering $\phi = \{\phi_0, \phi_1, \ldots\}$, is nonrecursive. It may be possible, however, to approximate K_ϕ by recursive sets. We note several results indicating that the degrees of recursive approximability of halting sets in arbitrary Gödel numberings have wide variation, while restriction to "optimal Gödel numberings" only narrows the possibilities slightly.

The original motivation for this work was the following type of problem. We know, for example, that the predicate calculus is undecidable, i.e., there is no total recursive procedure which tells whether a given formula is valid. Is it possible, however, that the problem may be decidable on a large fraction of its domain? Analogous questions may be asked about theories which are decidable, but only by very complex decision procedures. Can the decision problems be solved by fast procedures on large fractions of their domains?

These particular questions appear to be very notation-dependent. We study a related problem formulated in purely recursion-theoretic terms, in the hope of obtaining sufficiently invariant results to apply to our particular problems. Namely, we study the possibility of approximating the halting set $K_\phi = \{x \mid \phi_x(x) \text{ converges}\}$ for any Gödel numbering $\phi = \{\phi_0, \phi_1, \ldots\}$ by recursive sets. We discover that there are strong reasons why invariance is impossible; we have the widest possible variation in degree of possible approximation, dependent on the particular Gödel numbering chosen. The reason is that it is possible to construct Gödel numberings "padded" with many extra indices for trivial programs.

The definition of an "optimal Gödel numbering" [1] is an attempt to restrict as much as possible the ability to waste information by padding the Gödel numbering. However, this notion also does not yield sufficient invariance, although the possible variations are not as wide as in the general case. Thus, it appears that the general recursion-theoretic approach will not shed light on the original problems, and further work on these problems will be very dependent on notational conventions. The reasons for lack of invariance in the general recursion-theoretic case are of interest in themselves, however.

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571/9/2-2

We use notations and definitions from [2]. In addition, "$| A |$" refers to the cardinality of set A. "$\lfloor x \rfloor$" indicates the largest integer not greater than x. "$\lceil x \rceil$" indicates the smallest integer not less than x.

DEFINITION. A Gödel numbering is a function from N onto the set of partial recursive functions of one variable such that, letting ϕ_i be the image of $i \in N$:

(1) $\lambda i, x[\phi_i(x)]$ is a partial recursive function of two variables,

(2) $(\exists s, \text{recursive})\,(\forall i, x, y)[\phi_i(\langle x, y \rangle) = \phi_{s(i,x)}(y)]$.

We define the notion of density we will use in our approximations.

DEFINITION. If A is any set and r is any number, we say "dens$(A) < r$ a.e." (the density of A is less than r almost everywhere) if for all but finitely many n, $| A \cap \{0,..., n - 1\}|/n < r$; "dens$(A) < r$ i.o." (the density of A is less than r infinitely often) if for infinitely many n, $| A \cap \{0,..., n - 1\}|/n < r$. Analogous definitions are used for $>$, $\leqslant$ or $\geqslant$ in place of $<$.

Our first result shows that there are Gödel numberings ϕ for which K_ϕ is very closely approximable by recursive sets, simply because K_ϕ can be made very sparse. We need a lemma.

LEMMA 1. *For any creative set C, there exists a Gödel numbering ϕ such that $C = K_\phi$.*

Proof. Consider any Gödel numbering $\alpha = \{\alpha_0, \alpha_1, ...\}$. There exists f, a recursive 1–1 onto function, such that [2, Sect. 11.3]

$$x \in C \Leftrightarrow f(x) \in K_\alpha .$$

Define a new numbering ϕ by $\phi_i(x) = {}_{\alpha_{f(i)}}(f(x))$. Each ϕ_i is surely a partial recursive function. Using the fact that f and its inverse are total recursive, it is easy to show that every partial recursive function is represented among the ϕ_i's. $\lambda i, x[\phi_i(x)]$ is obviously partial recursive (property (1)). Finally, property (2) is demonstrated using the same property for α, together with the easy fact that for any partial recursive f, there is a total recursive g such that

$$(\forall i, x)\; \alpha_i(f(x)) = \alpha_{g(i)}(x).$$

Thus, ϕ is a Gödel numbering. Then $C = K_\phi$. ∎

PROPOSITION 1. *There is a Gödel numbering ϕ and recursive sets A and B, $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$, such that*

$$(\forall \epsilon > 0)[\text{dens}(A \cup B) > 1 - \epsilon \text{ a.e.}].$$

Proof. Let C be a creative set. Let $D = \{2^x \mid x \in C\}$. We claim that D is also creative, since it is clearly recursively enumerable and $C \leqslant_m D$ (see [1, Sect. 7.3]).

By Lemma 1, we may obtain a Gödel numbering ϕ with D as its halting set. Let $A = \varnothing$ and $B = N - \{2^x \mid x \in N\}$. ▮

We now wish to use similar ideas to show that there are Gödel numberings in which the halting set is *not* closely approximable by recursive sets. Two lemmas are needed. The first yields a "sparse" simple set.

LEMMA 2. *There exists a simple set S such that*

$$(\forall \epsilon > 0)[\text{dens}(S) < \epsilon \text{ a.e.}].$$

Proof. We construct S according to the following procedure. Dovetail all computations $\phi_i(x)$, for all i and x. Put x into S if $(\exists i)[\phi_i(x)$ converges and $x > 2^i$ and i is not *cancelled*]. *Cancel* i and continue.

S is easily shown to be simple. Only a single element x can be put into S for each index i, and by the lower bound on x at most $\lceil \log_2 n \rceil$ elements $\leqslant n$ are in S, so the density condition is satisfied. ▮

The second lemma yields a "nonapproximable" creative set.

LEMMA 3. *There is a creative set C such that*

$$(\forall A, B \textit{ recursive}, A \subseteq C \textit{ and } B \subseteq \bar{C})(\forall \epsilon > 0)[\text{dens}(A \cup B) < \epsilon \text{ a.e.}].$$

Proof. Let D be any creative set, and let S be the simple set constructed in Lemma 2. Define C as follows.

$$C = \{2^x \mid x \in D\} \cup \{y\text{th largest element of } N - \{2^x \mid x \in N\} \mid y \in S\}.$$

That is, C is equivalent to D on the arguments which are powers of two, and C is equivalent to S on the other arguments. C is creative since it is recursively enumerable and $D \leqslant_m C$.

If $k \in N - \{0\}$, then $\text{dens}(C) < 1/2k$ a.e., since the construction of C insures that the elements of C are a sparse subset of N. Thus, if A is recursive and $A \subseteq C$, we have $\text{dens}(A) < 1/2k$ a.e.

Assume B is recursive and $B \subseteq \bar{C}$. Since S is simple, it is easily shown that $B \cap (N - \{2^x \mid x \in N\})$ is finite. Thus, $B \subseteq \{2^x \mid x \in N\} \cup$ some finite set. But then, for any $k \in N - \{0\}$, $\text{dens}(B) < 1/2k$ a.e.

Combining these two facts gives $\text{dens}(A \cup B) < 1/k$ a.e., which (since k is arbitrary) yields the desired result. ▮

Lemma 3 answers a question posed by Albert Meyer in [3].

We may now use C to obtain our nonapproximable K_ϕ.

PROPOSITION 2. *There is a Gödel numbering ϕ such that*

$$(\forall A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)(\forall \epsilon > 0)[\text{dens}(A \cup B) < \epsilon \text{ a.e.}].$$

Proof. Immediate by Lemma 1. ∎

Similar constructions allow us to obtain intermediate results, Gödel numberings ϕ in which K_ϕ is approximable exactly to any desired rational between 0 and 1.

PROPOSITION 3. *For any rational r, $0 \leqslant r \leqslant 1$, there is a Gödel numbering ϕ such that*

$$(\exists A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)(\forall \epsilon > 0)$$
$$[\text{dens}(A \cup B) > r - \epsilon \text{ a.e.}],$$

and

$$(\forall A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)(\forall \epsilon > 0)$$
$$[\text{dens}(A \cup B) < r + \epsilon \text{ a.e.}].$$

Proof. We have already proved the result for $r = 0$ and $r = 1$. Assume $r \neq 0$, $r \neq 1$ and write r in the form a/b, where a and b are positive integers. We consider the set C defined in Lemma 3, and define a new set D as follows.

$$C_D(x) = \begin{cases} 0 & \text{if } (\exists y)[0 \leqslant y < a \text{ and } x \equiv y \bmod b], \\ C_C(\lfloor x/b \rfloor) & \text{otherwise.} \end{cases}$$

As before, D is creative.

Let ϕ be a Gödel numbering having $D = K_\phi$, as is possible by Lemma 1. Then $A = \varnothing$, $B = \{x \mid (\exists y)\ 0 \leqslant y < a \text{ and } x \equiv y \bmod b\}$ satisfy the first condition.

If $k \in N - \{0\}$, then $\text{dens}(K_\phi) < 1/2k$ a.e., since $\text{dens}(C) < 1/2k$ a.e. Thus, if A is recursive and $A \subseteq K_\phi$, we have $\text{dens}(A) < 1/2k$ a.e.

Assume B is recursive and $B \subseteq \bar{K}_\phi$, and $k \in N - \{0\}$. For any y, $0 \leqslant y < a$, $\text{dens}(B \cap \{x \mid x \equiv y \bmod b\}) < (1/b) + (1/2kb)$ a.e. For any y, $a \leqslant y < b$,

$$\text{dens}(B \cap \{x \mid x \equiv y \bmod b\}) < 1/2kb \text{ a.e.}$$

(since $B \cap \{x \mid x \equiv y \bmod b\}$ corresponds in a natural way to a recursive subset of $\bar{C}$.) Thus, $\text{dens}(B) < (a/b) + (1/2k)$ a.e.

Combining these two facts gives $\text{dens}(A \cup B) < (a/b) + (1/k)$ a.e., so (since k is arbitrary) $\text{dens}(A \cup B) < r + \epsilon$ a.e., as desired. ∎

Remark. With a little work, the arbitrary rational in Proposition 3 may be replaced by an arbitrary computable real number (i.e., a real number in which the successive digits may be effectively generated).

Remark. Similar techniques allow us to conclude the existence of a Gödel numbering ϕ such that

$$(\exists A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)(\forall \epsilon > 0)[\text{dens}(A \cup B) > 1 - \epsilon \text{ i.o.}],$$

and

$$(\forall A, B \text{ recursive}, A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)(\forall \epsilon > 0)[\text{dens}(A \cup B) < \epsilon \text{ i.o.}].$$

Propositions 1–3 and the above Remarks describe an "anything-goes" situation. A somewhat different set of possibilities exists if we restrict ourselves to the "optimal Gödel numberings" of Schnorr [1].

Definition. A Gödel numbering ϕ is called "optimal" if for *any* Gödel numbering $\alpha = \{\alpha_0, \alpha_1, ...\}$, there is a 1–1 total recursive function f and a constant c such that

$$(\forall i)[\alpha_i = \phi_{f(i)} \text{ and } f(i) \leqslant ci].$$

This condition on Gödel numberings is an attempt to obtain sharper results about computability by ruling out artificial Gödel numberings containing many extra indices for certain functions. It insures that all functions have indices not much larger than their indices in any other Gödel numbering.

For optimal numberings, we obtain results different from before, as not every creative set is the halting set for an *optimal* Gödel numbering. We obtain weakened versions of Propositions 1–3 and show that at least Propositions 1 and 2 are false in their full generality, for optimal Gödel numberings.

Proposition 4. *For any $\epsilon > 0$, there exists an optimal Gödel numbering ϕ and recursive sets A and B, $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$, such that* $\text{dens}(A \cup B) > 1 - \epsilon$ a.e.

Proof. We begin with an arbitrary optimal Gödel numbering $\alpha = \{\alpha_0, \alpha_1, ...\}$ and intersperse its indices with sufficiently many indices for a function such as $\lambda x[0]$, while retaining optimality. Specifically, for fixed ϵ we choose $k \in N$ such that $1/k < \epsilon$, and define

$$\phi_i(x) = \begin{cases} \alpha_{i/2k}(x) & \text{if } i \text{ is a multiple of } 2k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $\phi = \{\phi_0, \phi_1, ...\}$ is an optimal Gödel numbering. If we let $A = \{x \mid x \not\equiv 0 \bmod 2k\}$ and $B = \varnothing$, we have the required result. ∎

Proposition 5. *For any $\epsilon > 0$, there exists an optimal Gödel numbering ϕ such that*

$$(\forall A, B \textit{ recursive}, A \subseteq K_\phi \textit{ and } B \subseteq \bar{K}_\phi)[\text{dens}(A \cup B) < \epsilon \text{ a.e.}].$$

Proof. As in the proof of Proposition 4, we begin with an arbitrary optimal Gödel numbering $\alpha = \{\alpha_0, \alpha_1, ...\}$, and we also use the set C whose existence was proved in Lemma 3. We consider a fixed ϵ and choose $k \in N$ such that $1/k < \epsilon$.

Define

$$\phi_i(x) = \begin{cases} \alpha_{i/2k}(x) & \text{if } i \text{ is a multiple of } 2k, \\ 0 & \text{if } i \text{ is not a multiple of } 2k \text{ and } \lfloor i/2k \rfloor \in C, \text{ undefined otherwise.} \end{cases}$$

We claim that $\phi = \{\phi_0, \phi_1, ...\}$ has the needed properties. It is clearly an optimal Gödel numbering. If A and B are recursive, with $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$, then $\text{dens}((A \cup B) \cap \{x \mid x \equiv 0 \bmod 2k\}) < (1/2k) + (1/4k^2)$ a.e. The density is asymptotically less than or equal to $1/2k$, so for sufficiently large n, it is less than $(1/2k) + (1/4k^2)$. Also, if $0 < a < 2k$, $\text{dens}((A \cup B) \cap \{x \mid x \equiv a \bmod 2k\}) < 1/4k^2$ a.e. For if not, then A and B would yield new recursive sets A' and B' by $A' = \{x \mid 2kx + a \in A\}$, $B' = \{x \mid 2kx + a \in B\}$, which satisfy $A' \subseteq C$, $B' \subseteq \bar{C}$ and $\text{dens}(A' \cup B') > 1/2k$ i.o., a contradiction to the nonapproximability of C. Combining these facts, we obtain $\text{dens}(A \cup B) < 1/k$ a.e., so $\text{dens}(A \cup B) < \epsilon$ a.e., as required. ∎

In fact, we can show that the apparent weakness of Propositions 4 and 5 is inherent. It is impossible to obtain optimal Gödel numberings ϕ for which K_ϕ is arbitrarily approximable or nonapproximable. For this, we use as a lemma a result of Schnorr [1, p. 6].

LEMMA 4. *Let $\phi = \{\phi_0, \phi_1, ...\}$ and $\alpha = \{\alpha_0, \alpha_1, ...\}$ be optimal Gödel numberings. Then there exist a* 1–1 *onto recursive function f and a constant c such that*

$$(\forall i)[\alpha_i = \phi_{f(i)} \text{ and } f(i) \leqslant ci \text{ and } f^{-1}(i) \leqslant ci].$$

That is, any two optimal Gödel numberings are recursively isomorphic by a linear bounded isomorphism whose inverse is also linear bounded. We use this lemma in the following.

PROPOSITION 6. *All optimal Gödel numberings ϕ have the following two properties.*

(a) *There exists $\epsilon > 0$ and recursive sets A and B, with $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$, and* $\text{dens}(A \cup B) > \epsilon$ a.e.

(b) *There exists $\epsilon > 0$ such that for all recursive sets A and B, with $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$,* $\text{dens}(A \cup B) < 1 - \epsilon$ a.e.

Proof. (a) Assume we are given an optimal Gödel numbering ϕ. Apply the construction in the proof of Proposition 4 with $k = 1$ to obtain another optimal Gödel numbering $\phi' = \{\phi_0', \phi_1', ...\}$ and $A' = \{\text{odd integers}\}$, where $A' \subseteq K_{\phi'}$.

Then by Lemma 4 applied to ϕ and ϕ', we obtain f and c such that $(\forall i)[\phi_i' = \phi_{f(i)}$ and $f(i) \leqslant ci$ and $f^{-1}(i) \leqslant ci]$. Let $A = f(A')$, $B = \varnothing$. It is clear that $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$. Also, (for example), dens$(A \cup B) > 1/3c$ a.e., so $\epsilon = 1/3c$ satisfies the required condition.

(b) Again, assume we are given an optimal Gödel numbering ϕ. Fix any $k \in N - \{0\}$ and apply the construction in the proof of Proposition 5 to obtain a new optimal Gödel numbering $\phi' = \{\phi_0', \phi_1', ...\}$. Apply Lemma 4 and obtain f and c such that $(\forall i)[\phi_i' = \phi_{f(i)}$ and $f(i) \leqslant ci$ and $f^{-1}(i) \leqslant ci]$.

Consider any recursive $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi$. Assume that dens$(A \cup B) \geqslant 1 - 1/3kc$ i.o. If we let $A' = \{x \mid 2kx + 1 \in f^{-1}(A)\}$ and $B' = \{x \mid 2kx + 1 \in f^{-1}(B)\}$, we have $A' \subseteq C$ (since $x \in A'$ implies $f(2kx + 1) \in A \subseteq K_\phi$, which implies $\phi'_{2kx+1} \equiv 0$, by the isomorphism. But this means that $x \in C$, by the definition of ϕ'). Similarly, $B' \subseteq \bar{C}$. It is clear that A' and B' are recursive. Also, dens$(A' \cup B') \geqslant \frac{1}{4}$ i.o., for reasons which are "roughly" the following.

Since dens$(A \cup B) \geqslant 1 - 1/3kc$ i.o., it follows that for infinitely many n, out of the first $3kcn$ integers all but (approximately) n must be in $A \cup B$. For every $x \leqslant 3kn$, $f(x) \leqslant 3kcn$. Thus for the first $\frac{3}{2}n$ integers x such that $x \equiv 1 \bmod 2k$, we have $f(x) \leqslant 3kcn$. Since all but possibly n of these images under f must by in $A \cup B$, this means that (approximately) $\frac{1}{2}n$ of these $\frac{3}{2}n$ elements, or approximately $\frac{1}{3}$ of them, must have their images in $A \cup B$. We only conclude that dens$(A' \cup B') \geqslant \frac{1}{4}$ i.o. rather than dens$(A' \cup B') \geqslant \frac{1}{3}$ i.o., to outweigh the above approximations.

But these conditions on A' and B' contradict the definition of C. Thus we must have dens$(A \cup B) < 1 - 1/3kc$ a.e., and $\epsilon = 1/3kc$ satisfies the required condition. ∎

Finally, we may obtain the following analogy to Proposition 3, by methods similar to those used in proving Propositions 4 and 5.

Remark. For any rational r, $0 \leqslant r \leqslant 1$, and any $\epsilon > 0$, there exists an optimal Gödel numbering ϕ such that $(\exists A, B$ recursive, $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi)[$dens$(A \cup B) > r - \epsilon$ a.e.$]$, and $(\forall A, B$ recursive, $A \subseteq K_\phi$ and $B \subseteq \bar{K}_\phi)[$dens$(A \cup B) < r + \epsilon$ a.e.$]$.

Remark. For any $\epsilon > 0$, there exists an optimal Gödel numbering ϕ such that

$$(\exists A, B \text{ recursive, } A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)[\text{dens}(A \cup B) > 1 - \epsilon \text{ i.o.}],$$

and

$$(\forall A, B \text{ recursive, } A \subseteq K_\phi \text{ and } B \subseteq \bar{K}_\phi)[\text{dens}(A \cup B) < \epsilon \text{ i.o.}].$$

Open question. Can the first Remark be strengthened? Specifically, is Proposition 3 true for optimal Gödel numberings, provided $r \neq 0$, $r \neq 1$?

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