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On the automorphism group of certain algebraic varieties

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Contents

Abstract

We study the automorphism groups of two families of varieties. The first is the family of stable curves of low genus. To every such curve, we can associate a combinatorial object, a stable graph, which encode many properties of the curve. Combining the automorphisms of the graph with the known results on the automorphisms of smooth curves, we obtain precise descriptions of the automorphism groups for stable curves with low genera. The second is the family of numerical Godeaux surfaces. We compute in details the automorphism groups of numerical Godeaux surfaces with certain invariants; that is, corresponding to points in some specific connected components of the moduli space; we also give some estimates on the order of the automorphism groups of the other numerical Godeaux surfaces and some characterization on their structures.

0

Introduction

In almost every field of science there are a few very hard problems, whose solutions are often thought to be far in the future, or impossible at all. The usefulness of having a near-impossible problem lies both in its motivational nature and in the byproducts that the search can generate.

For algebraic geometry, the main problem is the classification of algebraic varieties. Its solution is satisfactory only for one dimensional varieties (curves), whereas already for surfaces the problem explodes and is possibly too vast to be tackled completely.

Nonetheless, mathematicians can try to focus on smaller problems directed towards the main goal. The study of automorphisms of algebraic varieties, or in other words of their symmetries is an example of this attitude. The presence of automorphisms with certain properties (for example of a certain order) or the property of having a certain automorphism group are indeed examples of restricting the huge problem of classification to a more concrete size. More concretely, we can use automorphisms to stratify the moduli space of algebraic varieties, providing locally closed subvarieties that can be descripted more precisely.

In 1893, Hurwitz applied the formula named after him to obtain an upper bound on the automorphism groups of algebraic curves (over the complex numbers) of genus at least 2, that is, on general type curves. The bound is based on the degree of the canonical divisor: $|\mathrm{Aut}(C)| \leq 42 \deg K_C$. There are curves attaining the Hurwitz bound for infinitely many genera, starting from 3, 7, and 14 (where we have the first occurrence of more than one Hurwitz curves with the same genus).

Restricting to characteristic zero and varieties of general type, we get that the automorphism group is always finite by Iitaka's Theorem (Theorem 6 in **[Iit77]**); a uniform bound, and even more a sharp bound, depending on the numerical properties of the canonical divisor is much harder to obtain.

For surfaces, several people improved upon the first exponential bound on the Chern classes proved by Andreotti in [**And50**], culminating in the sharp bound given by Xiao in [**Xia95**]: for a surface S of general type over the complex numbers we have $|\mathrm{Aut}(S)| \leq 42^2 K_S^2$. It is interesting to notice that this bound is attained exactly for products of Hurwitz curves.

Here we study the automorphism groups of two classes of algebraic varieties. In Chapter 1 we consider stable curves, defined by Deligne and Mumford to provide the natural object to add to the moduli space of curves to have a modular compactification. In Chapter 2 we examine the case of the surfaces of general type with the smallest invariants, numerical Godeaux surfaces: they are smooth surfaces of general type with $\mathrm{p_g} = 0$ and $K^2 = 1$.

For stable curves, we provide a tool to compute all the possible discrete data associated to it. These discrete data are encoded in a combinatorial object, called the stable graph. We will combine results on the automorphism groups of smooth curves with the automorphisms of stable graphs to give some results on the automorphisms of stable curves.

For numerical Godeaux surfaces, we provide two kind of results: on one hand, we provide the explicit stratification by automorphism groups for three connected components of the moduli spaces of such surfaces. On the other, we give some estimate on the number of automorphisms and on the structure of the automorphism groups for numerical Godeaux surfaces that have not yet been classified.

We refer to the introduction of Chapter 1 and 2 for more precise discussions on the results. Part of the results in these chapters have been published as [**MP11**] (in collaboration with Nicola Pagani) and [**Mag10**], respectively.

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1

Stable curves

If we have in mind the problem of the classification of algebraic varieties, it makes sense to start from the first non-trivial case: whereas zero-dimensional varieties are quite simple, one-dimensional varieties already pose several problems.

The first proponent of a "space of moduli" (parameters) of geometric object, and in particular of algebraic curves, was Riemann in [**Rie57**], who also started the study of their properties. Nonetheless, for about a century the concept of moduli space was used in a rather imprecise way, and only in the 1960s Mumford developed the necessary theory for a proper definition and construction of moduli spaces: geometric invariant theory [**Mum94**].

Well before the rigorous definition of the moduli space of smooth curves (of genus g), denoted with M_g, it was obvious that it had too few points to obtain a nice variety to play with. In terms of the curves represented in the moduli space, this is depicted by the fact that there are deformations of smooth curves that do not admit a smooth curve as their limit. Going back to the moduli space, this same fact means that the moduli space of smooth curve is not proper.

The intuitive solution of adding to the moduli space points that parametrize all possible deformations of smooth curves does not work too, because we would add far too many curves. For example, if a "one-parameter" curve C (a curve over a one-dimensional variety) deforms to a curve D with a rational tail (that is, with a rational component R intersecting transversally the rest of D in one point), then it can deforms also to $\overline{D \setminus R}$ (D with the rational tail contracted). In the moduli space, this example would give two non-separate points corresponding to D and to $\overline{D \setminus R}$.

A choice has to be made then, of singular curves to be included in the moduli space and ones to be excluded. The theoretical result that allows to choose is called the stable reduction theorem for curves [**DM69**]. It states that a family of smooth curves over the punctured disk can, possibly after a finite base change, be completed in a fundamentally unique way adding a central fiber in the class of stable curves, defined explicitly also in [**DM69**]. Indeed, this formulation is equivalent to stating that the valuative criterion of properness holds for the moduli *stack* of stable curves. Therefore we have a modular compactification of M_g, that we denote by $\overline{M}_g$.

Deligne and Mumford's compactification of M_g is far from being the only possibility; for a review on different options, linked to the minimal model program for M_g, see [**FS10**]. Nonetheless, $\overline{M}_g$ is by far the most used and important compactification, and, from a human point of view, the most natural, as it adds curves with the simplest singularities.

In the following section we will define precisely stable curves. For the moment being, we only observe that a stable curve is reduced but not irreducible, and its topological type depends only on the genera of the irreducible components and on which other components they intersect (as the intersections are by definition always transversal). Such information can be effectively encoded in a (decorated) graph, called the dual graph of the curve. In turn, a dual graph of a stable curve is called a stable graph.

Given an automorphism of a stable curve, we can observe how it acts on the dual graph. On the other hand, given an automorphism of the dual graph, there is always an automorphism of the curve inducing it. Hence, the automorphism group of a stable curve is composed of two parts: the automorphism group of the dual graph and the subgroup of automorphisms inducing the identity on the dual graph. These groups fit in a short exact sequence of finite groups, and we can use this short exact sequence to infer information on the automorphisms of stable curves from facts about the automorphisms of smooth curves and graphs.

Once the importance of having a proper moduli space has been acknowledged, it follows naturally the necessity to study a more general moduli problems for algebraic curves. Indeed, if we look at stable curves, that form the boundary of $\overline{M}_g$, we see that, as we anticipated, they can be reconstructed from the stable graph (that shows how the components intersect) and from a list of smooth curves (one for each vertex of the stable graph) with special points marked on them, corresponding to the points where they intersect the other components. The more general moduli problem is then the one of smooth curves with marked points, and it also has a nice compactification. These moduli space, denoted with $M_{g,n}$ and $\overline{M}_{g,n}$, where g is the arithmetic genus of the classified curves and n is the number of marked points, have been defined and firstly studied by [**Knu83**].

In section 1 we recall the precise definitions of the objects we are going to consider: stable curves and stable graphs. In section 2 we describe an algorithm to compute efficiently all possible stable graphs (of given invariants); finally, in section 3 we apply (and enhance) the algorithm to describe the automorphism groups of stable curves.

1.1 Preliminaries

1.1.1 *Stable curves*

We start giving the precise definitions of stable curve and stable pointed curves. These objects form respectively the (Deligne-Mumford) compactification of the moduli space of smooth curves and of smooth pointed curves.

1.1.1. Definition. A *prestable curve* is a reduced, proper, connected algebraic curve, with only smooth or node points.

1.1.2. Definition. A *stable curve* is a prestable curve with a finite automorphism group. Equivalently, a prestable curve such that:

- the normalization of an irreducible component of genus 0 contains at least three points in the preimage of the nodes of the curve;
- the normalization of an irreducible component of genus 1 contains at least one point in the preimage of the nodes of the curve.

Note that nodes (ordinary double points) contained in a component count as two points toward its stabilization.

1.1.3. Definition. A *stable pointed curve* is a prestable curve with the additional data of an ordered list of smooth points such that the automorphism group (as a pointed curve is finite). Equivalently, such that:

- the normalization of an irreducible component of genus 0 contains at least three points in the preimage of the union of nodes and marked points of the curve;
- the normalization of an irreducible component of genus 1 contains at least one point in the preimage of the union of nodes and marked points of the curve.

We can recover the arithmetic genus of a stable curve from the genus of its irreducible components and the number of nodes:

$$\mathrm{p_a}(C) = \#\text{nodes} - (K-1) + \sum_{i=0}^{K-1} g(\widetilde{C}_i)\,,$$

where $C_0, \ldots, C_{K-1}$ are the irreducible components of C, and $\widetilde{C}_i$ is the normalization of C_i.

From this remark, we find easily that there are no stable pointed curves of genus G, with N marked points whenever $2G-2+N \leq 0$, that is when (G,N) is $(0, \leq 2)$ or $(1,0)$. From now on, we fix G and N such that $2G-2+N > 0$

1.1.4. Notation. For K a positive integer, we define $\underline{K} = \{0, \ldots, K-1\}$ and Σ_K to be the symmetric group on the set $\underline{K}$. We will also write "stable curves" meaning "stable curves or stable pointed curves". We write *special points* of a stable curve for the preimage in its normalization of the union of nodes and marked points.

1.1.2 *Stable graphs*

If we look at how a stable curves C varies, we see that there is a discrete part and a continuous part. The latter is composed of the choice of the particular irreducible curves in in the moduli space of smooth curves of that genus, and on the choices of the points of that curve that intersect the rest of C or become self-intersections or marked points. The discrete part is the description of the number of irreducible components, their genera, the number of marked points for each component, and the intersections among components.

If we factor all the discrete data in a single object, we end up with the following definitions.

1.2.1. Definition.

- An *undirected multigraph* $\mathcal{G}$ is a couple (V,E) with V a finite set of *vertices* and E a finite multiset of *edges* with elements in $V \times V/\Sigma_2$.
- The multiplicity of the edge (v,w) in the multiset E is denoted by $\mathrm{mult}(v,w)$.
- The *total multiplicity* of $\mathcal{G}$, or its *number of edges*, is $|E|$: the cardinality of E as a multiset.
- The *degree* of a vertex v is defined as $\deg v := 2\,\mathrm{mult}(v,v) + \sum_{w\neq v} \mathrm{mult}(v,w)$.
- A *colored undirected multigraph* is a multigraph with some additional data attached to each vertex.

1.2.2. DEFINITION. A *stable graph* of type (G, N) is a colored undirected multigraph $\mathcal{G} = (V, E)$, subject to the following conditions.

(1) The color of a vertex v is given by a pair of natural numbers (g_v, n_v). The two numbers are called respectively the *genus* and the *number of marked points* of the vertex v.
(2) $\mathcal{G}$ is connected.
(3) Its *total genus*, defined as $\sum_{v \in V} g_v + |E| - (|V| - 1)$, equals G.
(4) Its *total number of marked points*, defined as $\sum_{v \in V} n_v$, equals N.
(5) Stability condition: $\deg v + n_v \geq 3$ for every vertex v with $g_v = 0$.

1.2.3. NOTATION. The number $\deg v + n_v$ is often called the *number of half edges* associated to the vertex v. Condition 5 can be rephrased in: for every vertex v of genus 0, its number of half edges is at least 3.

Two stable graphs $\mathcal{G} = (V, E, g, n)$ and $\mathcal{G}' = (V', E', g', n')$ are *isomorphic* if there is a bijection $f \colon V \to V'$ such that:

- $\mathrm{mult}(v, w) = \mathrm{mult}(f(v), f(w))$ for every $v, w \in V$;
- $g_v = g'_{f(v)}$ and $n_v = n'_{f(v)}$ for every $v \in V$.

1.2.4. REMARK. Note that from the definition just given, we are working with an unordered set of marked points. In other words, stable graphs as in the given definition relate to the moduli space of stable, genus g curves with n unordered points, that is $\overline{M}_{g,n}/\Sigma_n$. This choice is justified by the fact that in some applications these objects are important too; moreover avoiding the ordering of the marked points reduce by a vast amount the number of stable graph with given genus and marked points and makes the computations more manageable.

1.1.3 *Program*

We are going to describe the automorphism groups of all stable curves in a certain $\overline{M}_{g,n}$, for as many g and n as possible. The ingredients are the automorphism groups of the smooth components of the stable curves, and the structure of the stable graph.

For the latter, we need to compute all possible stable graphs, and their automorphisms. In the next section we will consider the two problems separately.

1.2 Generating stable graphs

1.2.1 *Description of the algorithm*

In this section we describe the general ideas of our algorithm. Let us first introduce the notation we use in the program.

2.1.1. Notation. The set of vertices V will always be $\underline{K}$, so that vertices will be identified with natural numbers $i, j, \ldots$. The multiplicity of the edge between i and j will be denoted by $a_{i,j}$: the symmetric matrix a is called the *adjacency matrix* of the stable graph. For convenience, we will denote $l_j := a_{j,j}$: it is the vector whose elements are the number of loops at the vertex j. For simplicity, we will consider g_j, n_j, l_j, $a_{i,j}$ to be defined also for i or j outside $\underline{K}$, in which case their value is always assumed to be 0.

2.1.2. Remark. In the following, we assume $|V| > 1$ in order not to deal with degenerate cases. There are trivially $G + 1$ stable graphs of type (G, N) with one vertex. Indeed, if there is exactly one vertex, the choice of the genus uniquely determines the number of loops on it after Definition 1.2.2.

The program uses recursive functions to generate the data that constitute a stable graph. In order, it generates the numbers g_j, then the numbers n_j, l_j (the diagonal part of the matrix a), and finally, row by row, a symmetric matrix representing a.

When all the data have been generated, it tests that all the conditions of Definition 1.2.2 hold, in particular that the graph is actually connected and satisfies the stability conditions. Then it uses the software `nauty` **[McK12]** to check if this graph is isomorphic to a previously generated graph. If this is not the case, it adds the graph to the list of graphs of genus G with N marked points.

A priori, for each entry of g, n, l, and a the program tries to fill that position with all the integers. This is of course not possible, indeed it is important to observe here that each datum is bounded. From below, a trivial bound is 0, that is, no datum can be negative. Instead, a simple upper bound can be given for each entry of g by the number G, and for each entry of n by the number N. For l and a, upper bounds are obtained from G using the condition on the total genus (Condition 1.2.2).

These bounds are coarse: Section 2.3 will be devoted to proving sharper bounds, from above and from below. Also, we will make these bounds dynamical: for instance assigning the value $g_0 > 0$ clearly lowers the bound for $g_j, j > 0$. The improvement of these bounds is crucial for the performance of the algorithm. In any case, once we know that there are bounds, we are sure that the recursion terminates.

The algorithm follows this principle: we want to generate the smallest possible number of couples of isomorphic stable graphs. To do so, we generalize the idea that to generate a vector for every class of vectors of length K modulo permutations, the simplest way is to generate vectors whose entries are increasing. The program fills the data row by row in the matrix:

$$\left(\begin{array}{cccc} g_0 & g_1 & \cdots & g_{K-1} \\ n_0 & n_1 & \cdots & n_{K-1} \\ l_0 & l_1 & \cdots & l_{K-1} \\ \hline \bullet & a_{0,1} & \cdots & a_{0,K-1} \\ a_{1,0} & \bullet & \ddots & \vdots \\ \vdots & \ddots & \bullet & a_{K-2,K-1} \\ a_{K-1,0} & \cdots & a_{K-1,K-2} & \bullet \end{array}\right), \tag{1}$$

and generates only matrices whose columns are ordered. Loosely speaking, we mean that we are ordering the columns lexicographically, but this requires a bit of care, for two reasons:

- the matrix a needs to be symmetric; in the program we generate only the strictly upper triangular part;
- the diagonal of a need not be considered when deciding if a column is greater than or equal to the previous one.

Therefore, to be precise, we define a relation (*order*) for adjacent columns. Let us call c_{j-1} and c_j two adjacent columns of the matrix (1). They are said to be equivalent if $c_{j-1,i} = c_{j,i}$ for any $i \notin \{j-1+3, j+3\}$. If they are not equivalent, denote with i_0 the minimum index such that $i_0 \notin \{j-1+3, j+3\}$ and $c_{j-1,i_0} \neq c_{j,i_0}$. Then we state the relation $c_{j-1} < c_j$ if and only if $c_{j-1,i_0} < c_{j,i_0}$. We do not define the relation for non-adjacent columns. We say that the data are ordered when the columns are weakly increasing, that is if, for all j, either c_{j-1} is equivalent to c_j or $c_{j-1} < c_j$.

To ensure that the columns are ordered (in the sense we explained before), the program keeps track of *divisions*. We start filling the genus vector g in a non decreasing way, and every time a value g_j strictly greater than g_{j-1} is assigned, we put a division before j. This means that, when assigning the value of n_j, we allow the algorithm to start again from 0 instead of n_{j-1}, because the column c_j is already bigger than the column c_{j-1}.

After completing g, we start filling the vector n in such a way that, within two divisions, it is non decreasing. Again we introduce a division before j every time we assign a value n_j strictly greater than n_{j-1}. We follow this procedure also for the vector l.

Finally, we start filling the rows of the matrix a. Here the procedure is a bit different. Indeed even if for the purpose of filling the matrix it is enough to deal only with the upper triangular part, imposing the conditions that the columns are ordered involves also the lower triangular part. A small computation gives that the value of $a_{i,j}$ is assigned starting from:

$$\begin{cases} 0 & \text{if there are divisions before } i \text{ and } j \\ a_{i,j-1} & \text{if there is a division before } i \text{ but not before } j \\ a_{i-1,j} & \text{if there is a division before } j \text{ but not before } i \\ \max\{a_{i,j-1}, a_{i-1,j}\} & \text{if there are no divisions before } i \text{ or } j, \end{cases}$$

and we put a division before i if $a_{i,j} > a_{i-1,j}$ and a division before j if $a_{i,j} > a_{i,j-1}$.

We cannot conclude immediately that this procedure gives us all possible data up to permutations as in the case of a single vector. This is because the transformation that the whole matrix undergoes when a permutation is applied is more complicated: for the first three rows (the vectors g, n, l), it just permutes the columns, but for the remaining rows, it permutes both rows and columns. Indeed, to prove that the procedure of generating only ordered columns does not miss any stable graph is the content of the following section.

1.2.2 *The program generates all graphs*

We want to prove the following result.

2.2.1. Proposition. *The algorithm described in the previous section generates at least one graph for every isomorphism class of stable graphs.*

From now on, besides G and N, we also fix the number of vertices K, and focus on proving that the algorithm generates at least one graph for every isomorphism class of stable graphs with K vertices.

2.2.2. Notation. We have decided previously to encode the data of a stable graph in a $(K + 3 \times K)$ matrix $G := (g, n, l, a)$ (cfr. (1)). We denote by $\mathcal{A}$ the set of all such matrices, and by $\mathcal{M}$ the set of all

$(K+3\times K)$ matrices that are generated by the algorithm described in the previous section.

We can assume that the graphs generated by the algorithm are stable, since we explicitly check connectedness and stability. In other words, we can assume the inclusion $\mathcal{M} \subset \mathcal{A}$. Hence, in order to prove Proposition 2.2.1, we will show that every $G \in \mathcal{A}$ is in $\mathcal{M}$ up to applying a permutation of $\underline{K}$. The idea is to give a characterization (Lemma 2.2.5) of the property of being an element of $\mathcal{M}$.

Recall first that the algorithm generates only matrices whose columns are ordered, as described in Section 2.1. More explicitly, if $G=(g,n,l,a)\in\mathcal{A}$, then $G\in\mathcal{M}$ if and only if:

$$\begin{aligned}
&\forall (i,j)\colon i \notin \{j-1,j\},\\
&\qquad g_{j-1} > g_j \qquad \text{does not happen},\\
&\qquad n_{j-1} > n_j \quad \Rightarrow g_{j-1} < g_j\,,\\
&\qquad l_{j-1} > l_j \quad\ \Rightarrow g_{j-1} < g_j \vee n_{j-1} < n_j\,, \text{ and}\\
&\qquad a_{i,j-1} > a_{i,j} \ \Rightarrow g_{j-1} < g_j \vee n_{j-1} < n_j \vee l_{j-1} < l_j \vee\\
&\qquad\qquad\qquad\qquad \exists i' < i : i' \notin \{j-1,j\} \wedge a_{i',j-1} < a_{i',j}\,.
\end{aligned}$$

Let us call a piece of data g_j, n_j, l_j, or $a_{i,j}$ a *breaking position* if it does not satisfy the condition above. Observe that a matrix $G\in\mathcal{A}$ has a breaking position if and only if G is not an element of $\mathcal{M}$.

We now introduce a total order on the set $\mathcal{A}$ of matrices $G=(g,n,l,a)$. If G is such a matrix, let $v(G)$ be the vector obtained by juxtaposing the vectors g, n, l and the rows of the upper triangular part of a. For example, if

$$G = \left(\begin{array}{cccc} 0 & 0 & 2 & 0\\ 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ \hline \bullet & 1 & 1 & 1\\ 1 & \bullet & 2 & 1\\ 1 & 2 & \bullet & 0\\ 1 & 1 & 0 & \bullet \end{array}\right)$$

(with the same structure as (1)), then we define

$$v(G) := (0,0,2,0,\quad 1,1,0,1,\quad 0,0,0,0,\quad 1,1,1,\quad 2,1,\quad 0)\,.$$

2.2.3. DEFINITION. If $G,H\in\mathcal{A}$, we write $G \prec H$ if and only if $v(G)$ is smaller than $v(H)$ in the lexicographic order. In this case we say that the matrix G is smaller than the matrix H.

Note that this total order on the set of matrices must not be confused with the partial order described in Section 2.1. From now on we will always refer to the latter order on $\mathcal{A}$.

2.2.4. Remark. If $\sigma \in \Sigma_K$ is a permutation and $G = (g, n, l, a)$ is a graph, then we can apply σ to the entries of the data of G, obtaining an isomorphic graph. The action of σ on G is: $(g, n, l, a) \to (g', n', l', a')$ where $g'_j = g_{\sigma(j)}$, $n'_j = n_{\sigma(j)}$, $l'_j = l_{\sigma(j)}$ and $a'_{i,j} = a_{\sigma(i),\sigma(j)}$. We denote this new matrix by σG. We write $\sigma_{i,j}$ for the element of Σ_K that corresponds to the transposition of $i, j \in \underline{K}$.

Now we are able to state the characterization we need to prove Proposition 2.2.1.

2.2.5. Lemma. *Let $G \in \mathcal{A}$; then $G \in \mathcal{M}$ if and only if G is minimal in the set*

$$\left\{\sigma_{j-1,j} G \mid 0 < j < K\right\}.$$

with respect to the order given in Definition 2.2.3.

Proof. We will prove that G is not minimal if and only if there is a breaking position.

Assume there is at least one breaking position in G. If there is one in g, n, or l, it is trivial to see that transposing the corresponding index with the previous one gives a smaller matrix. If this is not the case, let $a_{i,j}$ be a breaking position such that $a_{i',j}$ is not a breaking position whenever $i' < i$ (the position (i, j) is the first breaking position of its column). We deduce that $g_{j-1} = g_j$, $n_{j-1} = n_j$, $l_{j-1} = l_j$, and that for all $i' < i$ not in $\{j-1, j\}$, we have $a_{i',j-1} = a_{i',j}$. Let $H := \sigma_{j-1,j} G$; the vectors g, n, and l (the first three rows) coincide in G and H.

- If $j > i$, the smallest breaking position is in the upper triangular part of a; it is then clear that $H \prec G$.
- If $j < i$, the smallest breaking position is in the lower triangular part; by using the symmetry of the matrix a we again obtain $H \prec G$ (see the right part of Figure 1).

Conversely, let j be such that $H := \sigma_{j-1,j} G \prec G$. Then consider the first entry (reading from left to right) of the vector $v(G)$ that is strictly bigger than $v(H)$. This is a breaking position. Notice that if it occurs in the matrix a (equivalently, in the last K rows), it is actually the first breaking position of its column. □

The proof of Proposition 2.2.1 follows arguing as in this example.

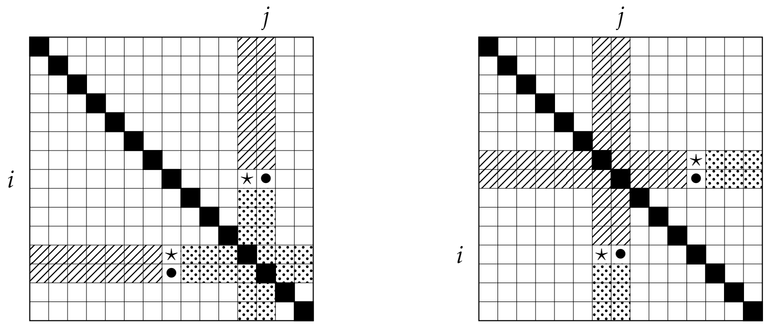


FIGURE 1. The matrix a when the first breaking position (the bullet) is $a_{i,j}$ with $j > i$ (left) or $j < i$ (right). When transposing $j-1$ and j, the white and the diagonal-filled entries do not change.

2.2.6. EXAMPLE. Let $G_0 := G \in \mathcal{A}$ be the graph of the previous example:

$$G_0 = \left(\begin{array}{cccc} 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline \bullet & 1 & 1 & 1 \\ 1 & \bullet & 2 & 1 \\ 1 & 2 & \bullet & 0 \\ 1 & 1 & 0 & \bullet \end{array}\right).$$

This graph is stable but not in $\mathcal{M}$ because, for example, $g_2 > g_3$ implies that g_3 is a breaking position. Thus we apply the permutation $\sigma_{2,3}$, obtaining the graph

$$G_1 := \sigma_{2,3} G_0 = \left(\begin{array}{cccc} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \bullet & 1 & 1 & 1 \\ 1 & \bullet & 1 & 2 \\ 1 & 1 & \bullet & 0 \\ 1 & 2 & 0 & \bullet \end{array}\right) \prec G_0 .$$

Now $a_{3,2}$ is a breaking position; applying $\sigma_{1,2}$, we obtain

$$G_2 := \sigma_{1,2} G_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \bullet & 1 & 1 & 1 \\ 1 & \bullet & 1 & 0 \\ 1 & 1 & \bullet & 2 \\ 1 & 0 & 2 & \bullet \end{array}\right) \prec G_1 .$$

This introduces a new breaking position at $a_{3,1}$, so we apply the transposition $\sigma_{0,1}$:

$$G_3 := \sigma_{0,1} G_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \bullet & 1 & 1 & 0 \\ 1 & \bullet & 1 & 1 \\ 1 & 1 & \bullet & 2 \\ 0 & 1 & 2 & \bullet \end{array}\right) \prec G_2 .$$

The graph G_3 is finally in $\mathcal{M}$ and indeed no transposition can make it smaller.

PROOF OF PROPOSITION 2.2.1. Recall that we have to prove that for every $G \in \mathcal{A}$, there is a permutation $\sigma \in \Sigma_K$ such that $\sigma G \in \mathcal{M}$.

So, let $G_0 = G \in \mathcal{A}$. If $G \in \mathcal{M}$, then we are done; otherwise, G does not satisfy the condition of Lemma 2.2.5, hence there is a transposition $\sigma_{j-1,j}$ such that $G_1 = \sigma_{j-1,j} G_0 \prec G_0$.

The iteration of this process comes to an end (that is, we arrive to a matrix in $\mathcal{M}$) since the set

$$\left\{\sigma G \mid \sigma \in \Sigma_K\right\}$$

is finite. $\square$

1.2.3 *Description of the ranges*

In Section 2.1 we have introduced the algorithm, by describing the divisions. In this section we introduce accurate ranges for the possible values of g, n, l and a.

We will deduce from the conditions of Definition 1.2.2 some other necessary conditions that can be checked before the graph is defined in its entirety. More precisely, every single datum is assigned trying all the possibilities within a range that depends upon the values of G and N, and upon the values of the data that have already been filled. The conditions we describe in the following are not the only ones possible; we tried other possibilities, but heuristically the others we tried did not give any improvement.

The order in which we assign the value of the data is g, n, l, and finally the upper triangular part of a row after row.

2.3.1. NOTATION. Suppose we are assigning the i-th value of one of the vectors g, n or l, or the (i,j)-th value of a. We define the following derived variables $e^{\max}$, c and p_1 that depend upon the values that have already been assigned to g, n, l, a.

We let $e^{\max}$ be the maximum number of edges that could be introduced in the subsequent iterations of the recursion, and c be the number of couples of (different) vertices already connected by an edge. We let p_1 be the number of vertices z to which the algorithm has assigned $g_z = 0$. Note that the final value of p_1 is determined when the first genus greater than 0 is assigned, in particular the final value of p_1 is determined at the end of the assignment of the values to the vector g. On the other hand, c starts to change its value only when the matrix a begins to be filled.

After the assignment of the i-th value, the derived values $e^{\max}$, c and p_1 are then updated according to the assignment itself.

2.3.2. NOTATION. When deciding g, n, or l, we let $n_i^{(2)}$ be the minimum between 2 and the number of half edges already assigned to the i-th vertex. This is justified by the fact that we know that, when we will fill the matrix a, we will increase by one the number of half edges at the vertex i in order to connect it to the rest of the graph. Hence, whenever $g_i = 0$, $n_i^{(2)}$ is the number of *stabilizing* half edges at the vertex i: one half edge is needed to connect the vertex to the rest of the graph, and then at least two more half edges are needed to stabilize the vertex. When deciding $a_{i,j}$, it is also useful to have defined h_i, the total number of half edges that hit the i-th vertex. Finally, we define

$$G_i := \sum_{i'<i} g_{i'}\,, \qquad N_i := \sum_{i'<i} n_{i'}\,,$$
$$N^{(2)} := \sum_{g_{i'}=0} n_{i'}^{(2)}\,, \qquad N_i^{(2)} := \sum_{\substack{i'<i\\ g_{i'}=0}} n_{i'}^{(2)}\,;$$
$$L_i := \sum_{i'<i} l_{i'}\,, \qquad A_{i,j} := \sum_{i'<i \vee j'<j} a_{i',j'}\,.$$

We are now ready to describe the ranges in which the data can vary. We study subsequently the cases of g, n, l and a, thus following the order of the recursions of our algorithm. Each range is described by presenting a first list of general constraints on the parameters and then by presenting a second list containing the actual ranges in the last line.

1.2.3.1 **Range for g_i.** When the algorithm is deciding the value of g_i, we have the following situation:

- $e^{\max} = G - G_i + K - 1$ by Condition 3;
- amongst the $e^{\max}$ edges, there are necessarily $K-1$ non-loop edges (to connect the graph); these $K-1$ edges give one half edge for each vertex, whereas we can choose arbitrarily where to send the other $K-2$ half edges; conversely, the $2(e^{\max} - K + 1)$ half edges of the remaining edges can be associated to any vertex; therefore, the maximum number of half edges (not counting those that are needed to connect the graph) is $2e^{\max} - K + N = 2(G - G_i) + K - 2 + N$;
- we need $2p_1$ half edges to stabilize the genus 0 vertices, since one half edge comes for free from the connection of the graph.

We use the following conditions to limit the choices for g_i:

(1) since g is the first vector to be generated, there is no division before i, hence

$$g_i \geq g_{i-1}\,;$$

remember that $g_j = 0$ whenever $j \notin \underline{K}$;

(2) we need at least $K-1$ non-loop edges, hence (using the fact that $\sum_{j\geq i} g_j \geq (K-i)g_i$)

$$\begin{aligned} & e^{\max} \geq K-1 \\ & \Rightarrow G - G_i - (K-i)g_i + K - 1 \geq K-1 \\ & \Rightarrow (K-i)g_i \leq G - G_i\,; \end{aligned}$$

(3) in order to stabilize the p_1 vertices of genus 0 (using the fact that one stabilizing half edge comes for free by connection) we must have

$$\begin{aligned} & 2p_1 \leq 2e^{\max} - K + N \\ & \Rightarrow 2p_1 \leq G - G_i - (K-i)g_i - K + N \\ & \Rightarrow (K-i)g_i \leq G - G_i - K + N - 2p_1\,. \end{aligned}$$

1.2.3.2 Range for n_i. When deciding n_i, we have the following situation:

- as before, $e^{\max} = G - G_K + K - 1 \geq K - 1$, and the maximum number of half edges still to be assigned is $2e^{\max} - K + N - N_i - n_i = 2(G - G_K) + K - 2 + N - N_i - n_i$;
- we need $2p_1 - N_i^{(2)} - n_i^{(2)}$ half edges to stabilize the first p_1 vertices;
- if $g_i = 0$, we need $2(i+1) - N_i^{(2)} - n_i^{(2)}$ more half edges to stabilize the first $i+1$ vertices.

The following conditions define then the ranges for the possible choices for n_i:

(1) if there is not a division before i (that is, if $g_i = g_{i-1}$), then we require $n_i \geq n_{i-1}$; otherwise, just $n_i \geq 0$;

(2) we cannot assign more than N marked points, hence (where we treat the case of $g_i = 0$ in a special way)

$$\begin{aligned} & N_i + n_i \leq N \\ & \Rightarrow n_i \leq N - N_i \\ & \Rightarrow (p_1 - i)n_i \leq N - N_i \;\text{ if moreover } g_i = 0. \end{aligned}$$

(3) if $g_i = 0$, for the purpose of stabilizing the first $i+1$ curves we cannot use marked points anymore, therefore we have

$$2(i+1) - N_i^{(2)} - n_i^{(2)} \leq \big(2(G - G_K) + K - 2\big)$$

$$\Rightarrow n_i^{(2)} = \min(2, n_i) \geq -\big(2(G - G_K) + K - 2\big) + \big(2(i+1) - N_i^{(2)}\big)$$

$$\Rightarrow \begin{cases} \text{impossible} & \text{if RHS} > 2 \\ n_i \geq \text{RHS} & \text{otherwise.} \end{cases}$$

1.2.3.3 **Range for l_i.** When deciding l_i, this is the situation:

- $e^{\max} = G - G_K - L_i - l_i + K - 1 \geq K - 1$, and the maximum number of half edges still to assign is $2e^{\max} - K = 2(G - G_K - L_i - l_i) + K - 2$;

The conditions on l_i are then the following:

(1) if there is not a division before i, then we require $l_i \geq l_{i-1}$; otherwise, just $l_i \geq 0$;

(2) we need at least $K - 1$ non-loop edges, hence

$$\begin{aligned} &e^{\max} \geq K - 1 \\ &\Rightarrow G - G_K - L_i - l_i + K - 1 \geq K - 1 \\ &\Rightarrow l_i \leq G - G_K - L_i\,; \end{aligned}$$

(3) let z be the index of the genus 0 vertex with the least number of stabilizing half edges such that $z < i$; it already has $n_z + 2l_z$ half edges, but we cannot use loops anymore to stabilize it; hence,

$$\begin{aligned} &\max(0, 2 - n_z - 2l_z) \leq G - G_K - L_i - l_i + K - 1 \\ &\Rightarrow l_i \leq G - G_K - L_i + K - 3 + n_z + 2l_z \end{aligned}$$

(4) assume $g_i = 0$; if $l_i > 0$, we are adding to the i-th vertex $2 - n_i^{(2)}$ stabilizing half edges, and to stabilize the p_1 genus 0 vertices, we need to have

$$2p_1 - N^{(2)} - \big(2 - n_i^{(2)}\big) \leq 2e^{\max} - K$$

$$\begin{aligned} \Rightarrow 2p_1 - N^{(2)} - \big(2 - n_i^{(2)}\big) \max(0, 2 - m_i) \leq \\ \leq 2(G - G_K - L_i - l_i + K - 1) - K \end{aligned}$$

$$\Rightarrow 2l_i \leq 2(G - G_K - L_i) + K + N^{(2)} - n_i^{(2)} - 2p_i\,.$$

(5) assume $g_i = 0$; after deciding l_i, we still have $e^{\max}$ edges to place, and each of them can contribute with one half edge

to the stabilization of the i-th vertex; moreover, one of these half edges is already counted for the stabilization; hence

$$\begin{aligned}&n_i + 2l_i + (e^{\max} - 1) \geq 2\\ &\Rightarrow n_i + 2l_i + G - G_K - L_i - l_i + K - 1 - 1 \geq 2\\ &\Rightarrow l_i \geq 4 - n_i - G + G_K + L_i - K\,.\end{aligned}$$

1.2.3.4 **Range for $a_{i,j}$.** When deciding $a_{i,j}$, this is the situation:

- earlier in Notation 2.3.2, we observed that for the purpose of filling the vectors g,n and l we could consider a genus 0 vertex stabilized when it had at least two half-edges (since the graph is going to be connected eventually). When assigning the values of a, the stability condition goes back to its original meaning, i.e. each vertex has at least 3 half edges.
- $e^{\max} = G - G_K - L_K - A_{i,j} + K - 1$;
- we have already placed edges between c couples of different vertices;

Here are the constraints that $a_{i,j}$ must satisfy:

(1) if there is not a division before i, then we require $a_{i,j} \geq a_{i-1,j}$; otherwise, just $a_{i,j} \geq 0$;
(2) if there is not a division before j, then we require $a_{i,j} \geq a_{i,j-1}$;
(3) we need at least $K - 2 - c$ (if positive) edges to connect the graph, because if $a_{i,j} > 0$, c will increase by 1 (this estimate could be very poor, but enforcing the connectedness condition in its entirety before completing the graph is too slow), hence:

$$\begin{aligned}&e^{\max} - a_{i,j} \geq \max(0, K - 2 - c)\\ &\Rightarrow a_{i,j} \leq G - G_K - L_K - A_{i,j} + K - 1 - \max(0, K - 2 - c)\,;\end{aligned}$$

(4) $a_{i,j}$ contributes with at most $\max(0, 3 - h_i) + \max(0, 3 - h_j)$ stabilizing half edges; hence, to stabilize the p_1 genus 0 vertices, we need

$$\begin{aligned}3p_1 - \sum_{g_{i'}=0} \min(3, n_i + 2l_i) - \big(\max(0, 3 - h_i) + \max(0, 3 - h_j)\big) \leq\\ \leq 2(e^{\max} - a_{i,j})\\ \Rightarrow 3p_1 - \sum_{g_{i'}=0} \min(3, n_i + 2l_i)\\ - \big(\max(0, 3 - h_i) + \max(0, 3 - h_j)\big) \leq\\ \leq 2(G - G_K - L_K - A_{i,j} + K - 1 - a_{i,j})\end{aligned}$$

$$\Rightarrow 2a_{i,j} \leq 2(G - G_K - L_K - A_{i,j} + K - 1) - 3p_1 + \sum_{g_{i'}=0} \min(3, n_i + 2l_i) + \max(0, 3 - h_i) + \max(0, 3 - h_j)\,.$$

(5) if $j = K - 1$ (that is, if this is the last chance to add half edges to the i-th vertex), then we add enough edges from i to $K - 1$ in order to stabilize the vertex i; moreover, if up to now we did not place any non-loop edge on the vertex i, we impose $a_{i,K-1} > 0$.

$$\begin{aligned} a_{i,K-1} &> 0 && \text{if } a_{i,j} = 0 \text{ for all } 1 < j < K - 1, \\ a_{i,K-1} &\geq 3 - h_i && \text{if } g_i = 0. \end{aligned}$$

1.2.4 *Performance*

The complexity of the problem we are trying to solve is intrinsically higher than polynomial, because already the amount of data to generate increases (at least) exponentially with the genera and the number of marked points. We also observed an exponential growth of the ratio between the time required to solve an instance of the problem and the number of graphs generated. Anyway, our program is specifically designed to attack the problem of stable graphs, and it can be expected to perform better than any general method to generate graphs applied to our situation.

We present here some of the results obtained by testing our program on an Intel® Core™2 Quad Processor Q9450 at 2.66 GHz. The version we tested is not designed for parallel processing, hence it used only one of the four cores available.

However, when computing a specific graph, the program needs to keep in the memory only the graphs with the same values in the vectors g, n, l. This allows us to neglect memory usage, but also shows that we can assign the computations of stable graphs with prescribed g, n, l to different cores or CPUs, thus having a highly parallelized implementation of the program.

In Table 1 we list, for each genus G, the maximum number of marked points N for which we can compute all the stable graphs of type (G, N) under 15 minutes.

In Figure 2 we show all the couples (G, N) that we computed against the time needed; the lines connect the results referring to the same genus. From this plot it seems that, for fixed G, the required time increases exponentially with N. However, we believe that in the long run the behaviour will be worse than exponential.

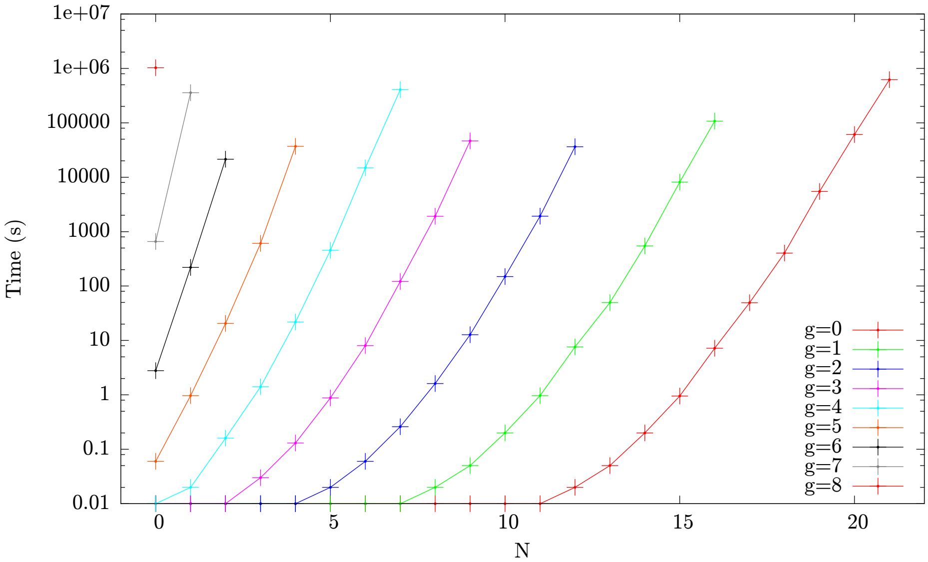


FIGURE 2. Logarithm of time needed to compute all stable graphs of type (G, N).

G	N	Time (s)	# stable graphs
0	18	392	847 511
1	14	539	1 832 119
2	10	147	1 282 008
3	7	117	1 280 752
4	5	459	2 543 211
5	3	606	2 575 193
6	1	226	962 172
7	0	681	1 281 678

TABLE 1. For small G, the maximum N such that all stable graphs of type (G, N) can be computed in less than 15 minutes.

More benchmarks and results are available at `boundary`'s webpage, `http://people.sissa.it/~maggiolo/boundary/`.

1.3 Computing automorphisms in low genera

We want to compute a stratification by automorphisms of the moduli spaces $\overline{M}_{g,n}$. To compute such a stratification using the stable graphs generated by the program described in the previous section, we need also to know the stratification for the moduli spaces of smooth pointed curves, $M_{g,n}$. Moreover, in some cases we need the stratification also for slightly more general objects. These objects are the moduli spaces of smooth curves of genus g with $n := \sum s_i n_i$ distinct marked points, grouped into two levels: first we have k ordered subsets, with the i-th containing s_i sets of n_i points. We denote these spaces by $M_{g,(n_1^{s_1},\dots,n_k^{s_k})}$.

For example, $M_{1,(2)} = M_{1,(2^1)}$ is the moduli space of smooth curves of genus 1 with two unordered marked points; $M_{0,(2^2)}$ is the moduli spaces of configurations of a 4-uple of points (p_1, p_2, p_3, p_4) in $\mathbb{P}^1$, where the configuration $(p_{i_1}, \dots, p_{i_4})$ is considered the same if and only if $\{i_1, i_2\}$ is $\{1, 2\}$ or $\{3, 4\}$.

More in general, the points of $M_{g,(n_1^{s_1},\dots,n_k^{s_k})}$ parametrize smooth curves C of genus g together with an object

$$
\begin{aligned}
(\{\{p_1, \dots, p_{n_1}\}, \dots, \{p_{(s_1-1)n_1+1}, \dots, p_{s_1 n_1}\}\}, \\
\dots, \\
\{\{p_{n-s_k n_k+1}, \dots, p_{n-(s_k-1)n_k}\}, \dots, \{p_{n-n_k+1}, \dots, p_n\}\}),
\end{aligned}
$$

with $p_i \in C$ all distinct, where $n := \sum s_i n_i$. Note that $M_{g,(1^n)} = M_{g,n}$, while $M_{g,(n)}$ is the moduli space of genus g curves with n unordered marked points, which can also be described as $[M_{g,n}/\Sigma_n]$. All other moduli spaces lies in between these two and are partial quotients for the same action.

We are going to compute stratifications for $\overline{M}_{1,i}$, with $i \in \{1, 2\}$, and for $\overline{M}_{2,0}$. To do these, we need the stratifications for the following moduli spaces:

- $M_{2,0}$, $M_{1,2}$, $M_{1,1}$ because they are the interior of the $\overline{M}_{g,n}$ we are studying,
- $M_{0,3}$, $M_{0,(1,2)}$, $M_{0,(1,1,2)}$, $M_{0,(2^2)}$, $M_{1,(2)}$, because, as we will show, they contribute to the boundary.

1.3.1 *Results for smooth curves*

It is a trivial fact that adding enough marked points to a curve makes it rigid. The precise number of marked points depends on the genus of the curve.

3.1.1. PROPOSITION. *A genus g curve with $n > 2g + 2$ marked points is rigid.*

Proof. Consider a smooth curve of genus g with an automorphism φ of order $k > 1$, and let D be the quotient of C by $\langle\varphi\rangle$. The morphism $C \to D$ is a finite ramified cyclic covering, and a marked point needs to be of total ramification. Hence, the contribution of n marked points to Hurwitz formula is $n(k-1)$, and the formula reduces to

$$2g - 2 = k(2g(D) - 2) + n(k-1) + Q\,,$$

where Q is the additional ramification. Since $Q \geq 0$, we have

$$n \leq \frac{2g - 2 - 2kg(D) + 2k}{k-1}$$

and the worst case for this bound occurs for $k = 2$ and $g(D) = 0$, yielding $n \leq 2g + 2$. $\square$

If we restrict to low genus, we may have non-trivial automorphisms only for curves in $M_{1,n}$ with $1 \leq n \leq 4$, and $M_{2,n}$ for $0 \leq n \leq 6$. Notice that in spite of this Proposition, stable curves may have non-trivial automorphisms even for n very large.

1.3.1.1 Stratification of $M_{0,\dots}$. It is well known that $M_{0,3}$ is a single point, whereas $M_{0,4}$ is $\mathbb{A}^3 \setminus \{0,1\}$. For $M_{0,(1,2)}$, we are allowed to take the involution fixing the first point and exchanging the other two, so it is isomorphic to $B\mathbb{Z}_2$.

The space $M_{0,(1,1,2)}$ is covered $2:1$ by $M_{0,4}$, sending $(0,1,\infty,x)$ to $(0,1,\{\infty,x\})$, and we note that starting from x or x^{-1} yields the same configuration in $M_{0,(1,1,2)}$. Therefore, there is only one involution to be added, for $x = -1$, and we have that $M_{0,(1,1,2)}$ is isomorphic to $[\mathbb{A}^1 \setminus \{0,1\}/(x \to x^{-1})]$.

Finally, $M_{0,(2^2)}$ has again a map from $M_{0,4}$; with a long but trivial check one prove that this map has maximum degree, 8, and that the automorphism group for the generic point is $\mathbb{Z}_2^2$, whereas the special point $\{\{0,\infty\},\{1,-1\}\}$ has automorphism group isomorphic to the dihedral group with 8 elements, D_4.

1.3.1.2 Stratification of $M_{1,n}$. From classical algebraic geometry, the generic point of $M_{1,1}$ has automorphism group equal to $\mathbb{Z}_2$, since a model of a point of $M_{1,1}$ is given by the cubic

$$y^2 = x^3 + ax + b$$

and we always have the involution sending (x,y) to $(x,-y)$. We have two special curves with automorphism group isomorphic to $\mathbb{Z}_4$ and to $\mathbb{Z}_6$: for $b = 0$ and $a = 0$ respectively.

To examine the strata of $M_{1,2}$, we need to start from elliptic curves in $M_{1,1}$ with an automorphism, and decide to add the second marked

Dimension	Name	Aut
2	$M_{1,2}$	$\{1\}$
1	A_2	$\mathbb{Z}_2$
0	P_4	$\mathbb{Z}_4$
0	P_6	$\mathbb{Z}_3$

Table 2. Stratification by automorphism groups of $M_{1,2}$. P_4 is contained in A_2, and the last three are contained in the first.

point on a fixed point of this automorphisms, otherwise trivially the automorphism will not "lift" to the point in $M_{1,2}$.

Therefore we need to study the fixed points of the automorphisms (omitting the point at infinity which is already the marked point of the elliptic curve). The involution on the generic cubic obviously fixes the three roots of the cubic polynomial, called the Weierstraß points. The automorphism of order 4 fixes only one of the Weierstraß points. The automorphism of order 6 fixes instead the points $(0, \pm 1)$.

If we start from a generic cubic and the involution, we have three choices for where to put the second marked point: the Weierstraß points. In the same way, the automorphism of order 4 lifts to $M_{1,2}$ since we can put the second marked point in the fixed point for both this automorphism and the involution. Instead, we cannot have an automorphism of order 6 because when we start with that one, there is not a common fixed point for both the involution and the automorphism of order 3. The automorphisms stratification is given in table 2. See also the part of figure 4 drawn with solid lines.

1.3.1.3 Stratification of $M_{1,(2)}$. This is the moduli space of genus 1 curves with two unordered marked points. The easiest way to compute the stratification is to think at $M_{1,2}$ and see if there can be more automorphisms for some special points. These automorphisms can come only from translations of the origin of the elliptic curve exchanging the two marked points.

If the second point p_2 on the elliptic curve is generic, the translation moving the origin to p_2 does not move back p_2 to the origin, hence the generic point of $M_{1,(2)}$ is still rigid: for having an additional automorphism, p_2 must be of 2-torsion inside the elliptic curve.

The space $A_2 \subset M_{1,2}$ consists indeed of an elliptic curve plus a 2-torsion point, hence in this locus the translation lifts to $M_{1,(2)}$. On the generic point of A_2, where the automorphism group is $\mathbb{Z}_2$, the

Dimension	Name	Aut
2	$M_{1,(2)}$	$\{1\}$
1	$A_{(2)}$	$\mathbb{Z}_2^2$
0	$P_{(4)}$	D_4
0	$P_{(6)}$	$\mathbb{Z}_3$

Table 3. Stratification by automorphism groups of $M_{1,(2)}$. $P_{(4)}$ is contained in $A_{(2)}$, and the last three are contained in the first.

Dimension	Name	Aut	Restrictions
3	$M_{2,0}$	$\mathbb{Z}_2$	None
2	T	$\mathbb{Z}_2^2$	$\gamma = \lambda(1-\mu)/(1-\lambda)$
1	T_1	$\mathbb{Z}_2 \times D_3$	$\mu = \lambda^{-1}(\lambda - 1), \gamma = 1/(1-\lambda)$
1	T_2	D_4	$\mu = \lambda^{-1}, \gamma = -1$
0	$T_{(1,2),1}$	$2D_6$	$(\lambda, \mu, \gamma) = (2, 1/2, -1)$
0	$T_{2,1}$	$\widetilde{\Sigma}_4$	$(\lambda, \mu, \gamma) = (i, -i, -1)$
0	R	$\mathbb{Z}_2 \times \mathbb{Z}_5$	$(\lambda, \mu, \gamma) = (1+\xi, \lambda+\xi^2, \mu+\xi^3)$

Table 4. Stratification by automorphism groups of $M_{2,0}$. $T_{(1,2),1}$ is contained in both T_1 and T_2, whereas $T_{2,1}$ is contained in T_2. All T_i are contained in T; T and R are contained in $M_{2,0}$. ξ is a fifth root of unity, $2D_6$ is a group with 24 elements, $\widetilde{\Sigma}_4$ is a 2-cover of Σ_4.

corresponding automorphism group in $M_{1,(2)}$ is easily computed to be $\mathbb{Z}_2^2$. On the special $\mathbb{Z}_4$ point, the translation does not commute and we obtain a D_4 point. The $\mathbb{Z}_3$ point in $M_{1,2}$ stays the same, as its second marked point is not of 2-torsion.

We can summarize these results in table 3. Note that the forgetful morphism $M_{1,2} \to M_{1,(2)}$ is a generic $2:1$ morphism ramified over the stratum A_2.

1.3.1.4 Stratification of $M_{2,0}$. This computation goes back to a paper of Bolza in 1887 [**Bol87**]. A more modern review of the result is in [**CGLR1999**], that we include here as table 4. A curve of genus 2 can always be written as

$$y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\gamma)\,;$$

the parameters restrictions refer to this model for the curve.

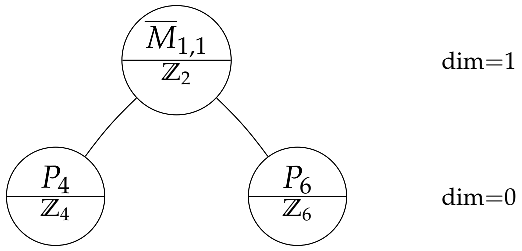
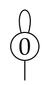


FIGURE 3. Hasse diagram for the strata determined by the automorphism group for $\overline{M}_{1,1}$.

1.3.2 *Automorphisms of stable curves*

In this section we combine the results about automorphisms of smooth pointed curves with the description of the stable graphs in the boundary of $M_{g,n}$ to provide a description of the stratification by automorphisms of $\overline{M}_{g,n}$ for low g and n. Results in the same direction can be found in **[Pag09]**.

1.3.2.1 Genus 1 curves. We start with the trivial case of $\overline{M}_{1,1}$. There is exactly one strata in the boundary, of dimension 0:

(2) 

that means a genus 0 curve with a self-intersection and one marked point. This graph has no automorphisms, so the automorphisms of the stable curves in this stratum are given by the automorphisms of the genus 0 component. Having one fixed point and two that can be swapped, this point is $M_{0,(1,2)}$. As computed in the previous section, it consists of a point with a $\mathbb{Z}_2$ automorphism group, as in the generic point of the open part of $M_{1,1}$. We can summarize the result as in figure 3.

For $\overline{M}_{1,2}$, we have four different stable graphs in the boundary, listed in table 5. Let us study for each of them the stratification by automorphisms.

- Case 1 is the degeneration obtained when the two marked points collide. The genus 0 curve is rigid, whereas the genus 1 curve provides a copy of $M_{1,1}$ on this stratum (in particular, the generic point has automorphism group $\mathbb{Z}_2$ and there are two special points with group $\mathbb{Z}_4$ and $\mathbb{Z}_6$).
- Case 2 happens instead when the curve degenerates to a singular one. The moduli space for this stratum is $M_{0,(1,1,2)}$,

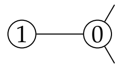
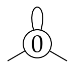
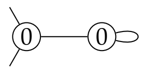
Id	Graph	dim	Aut($\mathcal{G}$)	Moduli
1		1	$\{1\}$	$M_{1,1} \times M_{0,3} \cong M_{1,1}$
2		1	$\{1\}$	$M_{0,(1,1,2)}$
3		0	$\{1\}$	$M_{0,(1,2)} \times M_{0,3} \cong M_{0,(1,2)}$
4		0	$\{1\}$	$M_{0,3} \times M_{0,(1,2)} \cong M_{0,(1,2)}$

TABLE 5. Stable graphs in the boundary of $\overline{M}_{1,2}$, with their strata's dimension, the automorphisms of the graph and the moduli corresponding to the strata. Case 3 is in the limit of case 2, whereas case 4 is in the limit of both case 1 and 2.

so as described before, it has only one non-rigid point, with automorphism group $\mathbb{Z}_2$.
- Case 3 happens on the boundary of the previous stratum, when the points defining the node collide. The first component is free to swap the two points of intersection with the second component, but once we use this fact, the second has three fixed points, justifying its description as $M_{0,(1,2)} \times M_{0,3}$; in particular, it consists of a single $\mathbb{Z}_2$ point.
- Case 4 is on the boundary of both case 1 and 2, and it should be clear that it consists of a single $\mathbb{Z}_2$ point.

Putting these cases together, we obtain that in the boundary there is one 1-dimensional stratum isomorphic to $\overline{M}_{1,1}$, consisting of cases 1 and 4, plus two isolated $\mathbb{Z}_2$ points: one as a special case of case 2 and one for case 3.

To obtain a comprehensive view of the whole $\overline{M}_{1,2}$, we should investigate if some of the automorphism strata in the boundary actually come from automorphism strata in the open part. For $\overline{M}_{1,2}$, we need to study the closure of A_2. But this is easy, as A_2 is the locus of genus 1 curves C with the choice of an origin and of a 2-torsion point (with respect to the origin). The elliptic curve degenerates to a nodal projective line, and there are two cases: if the marked point is the node (which is of 2-torsion) then we are in case 3, whereas if the node is not marked we have the $\mathbb{Z}_2$ point of case 2.

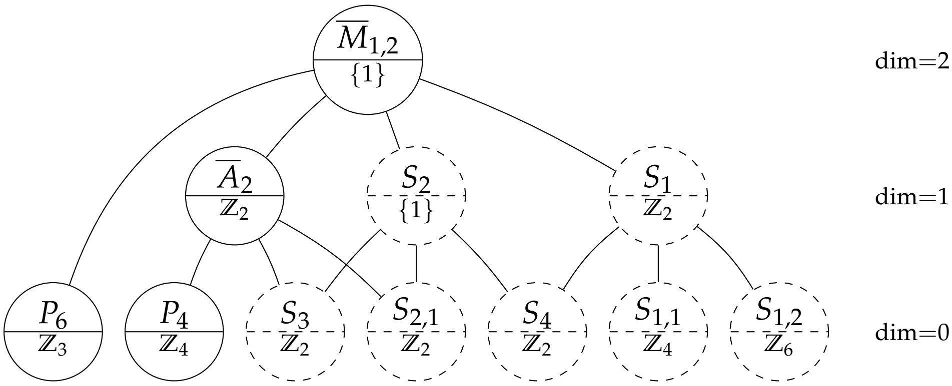


Figure 4. Hasse diagram including strata of $\overline{M_{1,2}}$ determined by the automorphism group or by the topology. A strata called S_i corresponds to stable graph of case i (with eventual special points inside them marked as $S_{i,j}$).

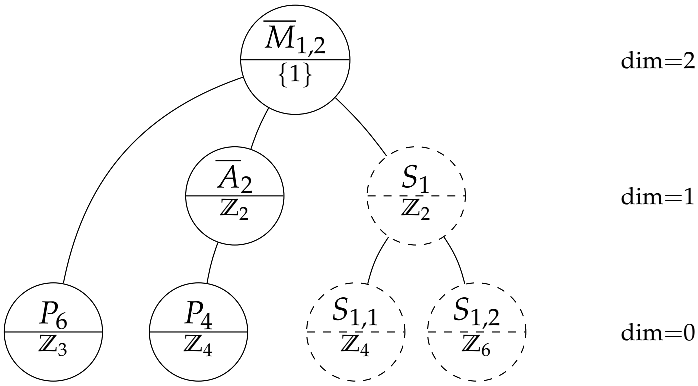


Figure 5. Hasse diagram for the strata determined by the automorphism group for $\overline{M}_{1,2}$. A strata called S_i corresponds to stable graph of case i (with eventual special points inside them marked as $S_{i,j}$).

3.2.1. Remark. Case 3 of table 5 has no group automorphisms because the marked points have a fixed order, hence the two vertices cannot be exchanged.

1.3.2.2 Genus 2 curves.

For $\overline{M}_{2,0}$, we have six different stable graphs in the boundary, listed in table 6. Let us study for each of them the stratification by automorphisms.

- Case 1 is the degeneration to a nodal curve of genus 1; this strata is a copy of $M_{1,(2)}$, since exchanging the two points identified in the node give the same curve.

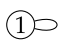
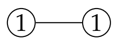
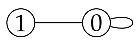
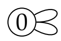
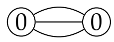
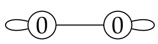
Id	Graph	dim	Aut($\mathcal{G}$)	Moduli
1		2	$\{1\}$	$M_{1,\{2\}}$
2		2	$\mathbb{Z}_2$	$M_{1,1} \times M_{1,1}$
3		1	$\{1\}$	$M_{1,1} \times M_{0,(1,2)}$
4		1	$\{1\}$	$M_{0,(2^2)}$
5		0	$\mathbb{Z}_2$	$M_{0,(3)} \times M_{0,3} \cong M_{0,(3)}$
6		0	$\mathbb{Z}_2$	$M_{0,(1,2)} \times M_{0,(1,2)}$

Table 6. Stable graphs in the boundary of $\overline{M}_{2,0}$, with their strata's dimension, the automorphisms of the graph and the moduli corresponding to the strata. Case 2 degenerates to case 3; case 1 to both cases 3 and 4. Moreover, case 3 degenerates to case 6 and case 4 to cases 5 and 6.

- Case 2 happens instead when the curve degenerates splits into two genus 1 curve intersecting in a point. The strata is then a product of $M_{1,1}$ by itself, quotiented by the involution of the graph that exchanges the two elliptic curves E_1 and E_2. This involution has fixed points only when the two curves are isomorphic, so we can list all the automorphism strata of case 2: the generic curve, has automorphism group $\mathbb{Z}_2^2$, corresponding to the two involutions of E_1 and E_2; we have then two 1-dimensional strata with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_6$ when one of the curve is more symmetric, and a point with automorphism group $\mathbb{Z}_4 \times \mathbb{Z}_6$ when the two curves are both special. Moreover, for $E_1 = E_2$, we obtain a 1-dimensional strata with automorphism group D_4, containing two points with automorphism group of order 32 and 72, corresponding to the two special curves.
- Case 3 happens on the boundary of the two strata 1 and 2. It is a copy of $M_{1,1} \times M_{0,(1,2)}$, hence its generic automorphism group is $\mathbb{Z}_2^2$, with two special points $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_6 \times \mathbb{Z}_2$.

- Case 4 is on the boundary of case 1, it is a copy of $M_{0,(2^2)}$, hence its generic automorphism group is $\mathbb{Z}_2^2$ with a special D_4 point.
- Case 5 is on the boundary of case 4 and consists of a single point with an automorphism group which is an extension of Σ_3 by $\mathbb{Z}_2$.
- Case 6 is on the boundary of cases 3 and 4 and consists of a single point with an automorphism group which is an extension of D_4 by $\mathbb{Z}_2$.

To obtain a stratification of $M_{2,0}$, one needs to study carefully the degenerations of the higher-dimensional automorphism strata to the boundary, to understand which boundary strata are irrelevant as consisting in the boundary of another strata. Nonetheless, the groups described in the previous list and in table 4 constitutes all the possible automorphism groups of a stable curve of genus 2.

2

Numerical Godeaux surfaces

We already saw how moduli space of curves are far from being a boring object of study. Nonetheless, we continue, exploring a bit the world of moduli spaces of algebraic surfaces, that is, the two-dimensional case.

In the case of smooth curves there is an obvious, that turns out to be also unique, discrete invariant that encodes the topological type of the curve, the genus. This means that M_g and M_h for $g \neq h$ are different and disjoint. The genus is particularly nice as a discrete invariant also because M_g is connected and irreducible; moreover, the Deligne-Mumford compactification adds a boundary to each M_g in such a way that $\overline{M}_g$ continue to be connected (of course) and disjoint from M_h with $g \neq h$.

restrict to is the analogous of having genus at least 2: we need to work with surfaces of general type. The moduli space for surfaces of general type, modulo birational equivalence, has been defined in [**Gie77**]. We have two obvious discrete invariants, namely the Euler characteristic χ and the self-intersection of the canonical divisor, K^2. These are not sufficient to get connected moduli spaces M_{χ,K^2}; instead, the number of their component can be arbitrarily large, even if finite. Each connected component can be also reducible and non-reduced. On the other hand, not all "reasonable" pairs (χ, K^2) correspond to non-empty moduli spaces. Compactification of moduli of surfaces (and even more for higher dimensional varieties) is a very complicated subject and, despite the theory being more or less settled since [**KSB88**], very few cases of explicit compactifications are known, for example the compactification of some connected components of the moduli spaces of Campedelli and Burniat surfaces in [**AP09**].

The situation being so complicated, it is almost always necessary to restrict to smaller classes of surfaces, for example fixing geometric genus and irregularity to some small value.

Numerical Godeaux surfaces are the algebraic surfaces of general type with the smallest invariants, $K^2 = 1$ and $\mathrm{p_g} = 0$. For this reason they have been studied thoroughly in the history of the classification of algebraic surfaces. Conjectured to be rational by Max Noether as a subclass of the surfaces with $p_g = 0$ and $q = 0$, they take their name from Lucien Godeaux: in 1931, he constructed one of them, providing the first example of minimal surface of general type with $p_g = 0$. This particular example is called *Godeaux surface*.

A first classification appears in [**Miy76**] by Miyaoka: numerical Godeaux surfaces are split in five classes up to their torsion group. In [**Rei78**], Reid constructs the moduli spaces of the three classes with larger torsion group. Up to now, even if several examples of surfaces in the other two classes are known, there are no similar constructions for them.

Recently, another viewpoint has been pursued: the observation that all sporadic examples of numerical Godeaux surfaces with small torsion group admit an involution led to the study of numerical Godeaux surfaces with an involution. This study has been completed by Calabri, Ciliberto and Mendes Lopes in [**CCML07**], who prove a classification theorem for such surfaces. The following step, classification of numerical Godeaux surfaces with an automorphism of order three, has been completed by Palmieri in [**Pal08**], who found that there are no such surfaces.

We consider two problems in this chapter. On one hand, the problem of finding all the automorphisms of the surfaces for which the moduli spaces is known, the ones with large torsion group. Using the constructions found by Reid, we are able to compute explicitly the automorphism groups of such surfaces. On the other hand, we work in generality to provide estimates on the number of automorphisms for a general numerical Godeaux surface, and possibly some insight on the structure of the automorphism group.

In section 1 we provide some information on the torsion group of numerical Godeaux surfaces, which is essential in their study. We also recall briefly the construction of the moduli spaces of numerical Godeaux surfaces with torsion of order 3, 4 or 5 done in [**Rei78**]. These notions are necessary to compute the automorphism groups of such surfaces, as we do in 2. There, we also organize the results in terms of strata of the moduli spaces. In section 3 we use general arguments about fibrations to provide estimates and insights on the structure of the automorphism groups of numerical Godeaux surfaces.

2.1 Classification and construction

2.1.1 *Preliminaries*

In this section we will recall briefly what numerical Godeaux surfaces are, how they can be classified into five classes depending on the their torsion group (following [**Miy76**]), and how to construct the coarse moduli space for numerical Godeaux surfaces with torsion isomorphic to $\mathbb{Z}_5$, $\mathbb{Z}_4$ and $\mathbb{Z}_3$ (following [**Rei78**]). Let us start with the raw definition.

1.1.1. Definition. A *numerical Godeaux surface* is a minimal smooth surface of general type S with $K_S^2 = 1$, $\mathrm{p_g}(S) = \mathrm{q}(S) = 0$ so that $\chi(\mathcal{O}_S) = 1$. For brevity, in the following we will write simply *Godeaux surfaces* for them.

1.1.2. Definition. The torsion group of an algebraic surface is the torsion part of its Picard group $\mathrm{Pic}(X)$, and it is denoted by $\mathrm{Tors}(X)$.

2.1.2 *Torsion group of Godeaux surfaces*

Consider a surface S for which $\mathrm{Tors}(S)$ is non-trivial, and take a non-zero torsion divisor D, of order, let us say, k. We can associate to these data, together with an isomorphism $\mathcal{O}_S(kD) \cong \mathcal{O}_S$, a connected cover of S, $\pi\colon X \to S$, where X is defined as the relative spectrum over S of the $\mathcal{O}_S$-algebra $\mathcal{O}_S \oplus \mathcal{O}_S(D) \oplus \cdots \oplus \mathcal{O}_S((k-1)D)$. The cover π is an étale cover with k sheets. More in general, this works even when $\mathrm{Tors}(S)$ is not cyclic, giving an étale cover with order of $\mathrm{Tors}(S)$ sheets.

If we are in such a situation, then by elementary facts $\chi(\mathcal{O}_X) = k\chi(\mathcal{O}_S)$ and $K_X^2 = kK_S^2$ (from $K_X = \pi^\star K_S$). In our situation, $K_S^2 = 1 = \chi(\mathcal{O}_S) = 1$, so $K_X^2 = k = \chi(\mathcal{O}_X)$. From Lemma 14 in [**Bom73**], if $\mathrm{q}(X) \geq 1$ we have $2\chi(\mathcal{O}_X) \leq K_X^2$, and since this is not the case here, we have $\mathrm{q}(X) = 0$. From this, we deduce $\mathrm{p_g}(X) = k-1$. Since the canonical divisor of X is the pullback of the one of S, that is nef and big, also K_X is nef and big and X is minimal.

Theorem 14 in [**Bom73**] gives that if k is the order of $\mathrm{Tors}(S)$, then $\mathrm{p_g}(S) \leq {}^1/_2 K_S^2 + {}^3/_k - 1$, hence $k \leq 6$ for a Godeaux surface.

1.2.1. Proposition. *For a Godeaux surface S, $|\mathrm{Tors}(S)| \leq 5$.*

Proof. If $|\mathrm{Tors}(S)| = 6$, the corresponding étale cover X would have $K_x^2 = 6$, $\mathrm{p_g}(X) = 5$ and $\mathrm{q}(X) = 0$; moreover, it would admit a free automorphism of order 6. Being $K_X^2 = 6 = 2\,p_g(X) - 4$, X is a Horikawa surface (see [**Hor81**]). In particular, we know that the canonical map is a morphism two to one onto a surface of degree $p_g(X) - 2$ in $\mathbb{P}^{p_g(X)-1}$. In our case, the image is a surface of degree 2 in $\mathbb{P}^4$: the Hirzebruch surface $\mathbb{F}_1$ embedded as a rational normal scroll. But $\mathbb{F}_1$ admits a fibration over $\mathbb{P}^1$ with fibers of genus 2. Since it is canonically defined, the fibration is G-invariant and we can apply Lemma 1.2.2; we get that 6 divides 1, hence such a surface X cannot exist. □

1.2.2. LEMMA. *Let X be a smooth, minimal surface of general type, with an automorphism α of order k such that all non-trivial α^i act freely on X. Let $f\colon X \to \mathbb{P}^1$ be a fibration with fibers of genus g, compatible with α. Then $k \mid g - 1$.*

PROOF. Let $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ commuting with f and α; β is represented by a matrix in $\mathbb{P}\,\mathrm{GL}(2)$, and has at least one fixed point, say, $x \in \mathbb{P}^1$. If F is the fiber over x, $F^2 = 0$; on the other hand, from the adjunction formula we get $K_X \cdot F = K_F \cdot F - F^2 = 2(g-1)$. If $S = X/\langle\alpha\rangle$, and $\pi\colon X \to S$ is the projection, we have $K_X = \pi^\star K_S$, and $F = \pi^\star F'$ for some $F' \in \mathrm{Pic}(S)$ (because F is α-invariant). Therefore, $2(g-1) = K_X \cdot F = \pi^\star K_X \cdot \pi^\star F' = k(K_S \cdot F')$. Since $\pi^\star F' = F$, we have $0 = F^2 = kF'^2$, and so by the genus formula $K_X \cdot F'$ is even. □

There is another case missing: the torsion isomorphic to $\mathbb{Z}_2^2$.

1.2.3. PROPOSITION. *For a Godeaux surface S,* $\mathrm{Tors}(S)$ *is cyclic.*

PROOF. Suppose that $\mathrm{Tors}(S) \cong \mathbb{Z}_2^2$; by Lemma 1.2.5, we have three non-zero sections $x_{i,j}$ with $i, j \in \{0,1\}$ and not both zero, in $\mathrm{H}^0(S, K_S + D_{i,j})$, where $D_{i,j}$ are all the torsion divisors of S. When we square them, we obtain three non-zero sections of $\mathrm{H}^0(S, 2K_S)$. This has dimension 2, hence we have a non-trivial relation between $x_{i,j}^2$.

If $\pi\colon X \to S$ is the ètale cover associated to $\mathrm{Tors}(S)$, we have $K_X^2 = 4$, $\chi(\mathcal{O}_X) = 4$, $q(X) = 0$, and so $p_g(X) = 3$.

The pullbacks $x'_{i,j}$ of $x_{i,j}$ to X are independent because in different eigenspaces inside $\mathrm{H}^0(X, K_X)$, so they form a basis; on the other hand, the relation pulls back to a relation in $\mathrm{H}^0(X, 2K_X)$.

Let $|K_X| = F + |M|$; the image of $\varphi_{|M|}$ lies in $\mathbb{P}^2$, and satisfies a quadric relation; it is therefore a conic; hence M is divisible by 2 and we can write $|K_X| = F + |2T|$ for some T.

Now, $4 = K_X^2 = K_X \cdot F + 2K_X \cdot T$. Since X is minimal of general type, $K_X \cdot F \geq 0$, and so $0 \leq K_X \cdot T \leq 2$. Suppose that $F = 0$: then $K_X \cdot T = 2 = 2T^2$, but by adjunction $K_X \cdot T = (K_T - T) \cdot T = 2(g(T) - 1) - T^2 = 2(g(T) - 1) - 1$ is odd. Hence F must be non-zero, that is, $x'_{i,j}$ (and also $x_{i,j}$) must have a common component. But this is not possible by Lemma 1.2.6. $\square$

1.2.4. LEMMA. *Let $D = \sum_{i \in I} m_i C_i$ a divisor numerically equivalent to zero on a smooth, minimal surface of general type such that $K_S \cdot C_i = 0$ for every $i \in I$; then $D = 0$.*

PROOF. Since S is of general type, K_S is nef and big; let $J \subseteq I$ the subset of indices such that $m_j < 0$ and define $D_1 := \sum_{i \in I \setminus J} m_i C_i$, $D_2 := \sum_{j \in J} -m_j C_j$. D_1 and D_2 are effective divisor, numerically equivalent, and without common components. We have $D_i^2 = D_1 \cdot D_2 \geq 0$; on the other hand, by the Index Theorem (Corollary 2.16 in [**BHPV04**]), $K_S^2 > 0$ implies $D_i^2 \leq 0$ with equality if and only if $D_i = 0$. $\square$

1.2.5. LEMMA. *Let S be a Godeaux surface; then for every divisor $D \in$ $\mathrm{Tors}(S)$ and for every $n \geq 1$ (apart from $n = 1$, $D = 0$), we have*

$$\mathrm{h}^0(S, nK_S + D) = 1 + \binom{n}{2}.$$

PROOF. See [**Rei78**], Lemma 0.4. $\square$

1.2.6. LEMMA. *Let S be a Godeaux surface and D_1, D_2 two distinct, effective divisors, numerically equivalent to K_S. Then, they do not have common components.*

PROOF. See [**Rei78**], Lemma 0.1. $\square$

Therefore, the torsion group of a Godeaux surface is cyclic of order at most 5. We will see the description of the moduli spaces of Godeaux surfaces with torsion isomorphic to $\mathbb{Z}_5$, Z_4 and $\mathbb{Z}_3$ as given in [**Rei78**]; nonetheless, Godeaux surfaces with $\mathbb{Z}_2$ or trivial torsion have been constructed, the first in chronological order were in [**Bar84**] and [**Bar85**].

As said, in the following sections, we will recall the construction of the moduli spaces of Godeaux surfaces with torsion of order at least 3; the main tool is the study of the universal Galois cover (that is constructed via the torsion group), and the relations between the canonical ring of a Godeaux surface and of its cover. To fix notations, from now on S will be a Godeaux surface and $\psi\colon X \to S$ the cover associated to its torsion group G.

2.1.3 *Torsion of order five*

This is a basic computation, since we can easily find the invariants of X and check that in particular $K_X^2 = 2\,\mathrm{p_g}(X) - 3$, that is X is a Horikawa surface. Then from **[Hor76]** we know that X is birational to a quintic hypersurface in $\mathbb{P} = \mathbb{P}(x_1, x_2, x_3, x_4)$, with $\deg x_i = 1$, by the canonical map $\varphi\colon X \to \overline{X} \subset \mathbb{P}$. Moreover, $\overline{X}$ has at most rational double points as singularities.

The group G acts naturally on X and on $\mathrm{H}^0(K_X)$; so $\mathrm{H}^0(K_X)$ is a G-module and we know that this G-module decomposes as the direct sum of the four nontrivial characters of G (see for example Proposition 2.4 in [**MLP08**]). Therefore we may assume that the action ϱ of G on $\mathbb{P}$ is fixed and generated by the automorphism $\mathrm{diag}(\xi, \xi^2, \xi^3, \xi^4)$, where ξ is a fixed primitive fifth root of unity. Moreover, we will often identify G with its image in $\mathrm{Aut}(\mathbb{P})$.

Hence we have to classify all quintic hypersurfaces $\overline{X} \subset \mathbb{P}$, fixed by this action, with at most rational double points. We will not specify explicitly the locus of quintics that do not satisfy the latter condition; as for the former, we only have to require that the monomials composing the equation of $\overline{X}$ are in the same eigenspace of $\mathrm{H}^0(X, 5K_X)$ with respect to the action of G. Since $\overline{X}$ cannot pass through the fixed points of the action (which are the coordinate points), in the equation there are necessarily the monomials x_i^5, so the eigenspace is fixed to be the one containing these monomials. Summing up, the equation is of this kind:

$$\begin{aligned} q_0 = {} & x_1^5 + x_2^5 + x_3^5 + x_4^5 \\ & + b_1 x_2 x_3^3 x_4 + b_2 x_1^3 x_3 x_4 + b_3 x_1 x_2 x_4^3 + b_4 x_1 x_2^3 x_3 \\ & + c_1 x_2^2 x_3 x_4^2 + c_2 x_1 x_3^2 x_4^2 + c_3 x_1^2 x_2^2 x_4 + c_4 x_1^2 x_2 x_3^2 . \end{aligned} \tag{3}$$

We have eight affine parameters; to obtain the coarse moduli space, we have to remove the points which give surfaces with singularities worse than rational double points and to quotient by isomorphisms of the correspondent Godeaux surfaces. Such an isomorphism lifts as an isomorphism of $\mathbb{P}$ which sends the first $\overline{X}$ in the second and commutes with ϱ. As we will see in Remark 2.1.3, for any surface there are only finitely many points corresponding to surfaces isomorphic to the first one; this means that we are quotienting by a finite group (this is true also for the next two cases). Its action is far from being free, nevertheless we have the following.

1.3.1. Theorem. *The coarse moduli space M_5 of Godeaux surface with torsion of order 5 is a finite quotient of a nonempty open subset $\tilde{M}_5$ of $\mathbb{A}^8$. A point*

$$(b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4)$$

corresponds to the Godeaux surface obtained resolving the singularities of the quotient by ϱ of the variety defined by equation (3) *in* $\mathbb{P}$.

2.1.4 *Torsion or order four*

For the next two cases, we will compute generators and relations of the canonical ring of X. We may use direct computation using the property of the canonical ring of S, or gain help from the numerator of the Hilbert series as noted in [**Rei00**]. For torsion of order 4, we need five generators: x_1, x_2, x_3 in degree 1, and y_1, y_3 in degree 2 (the subscripts denote the eigenspace in which the generators lie). So X is naturally embedded in the weighted projective space $\mathbb{P} = \mathbb{P}(1^3, 2^2)$. As for the relations, we have two of them in degree 4, q_0 and q_2 (again, the subscripts denote the eigenspaces). These generators and relations describe the canonical ring of X, and one proves that the bicanonical map $\varphi\colon X \to \overline{X} \subset \mathbb{P}$ is a birational morphism and $\overline{X}$ has at most rational double points. Again, we can fix the action ϱ of G (so that it is diagonal) on $\mathbb{P}$ and exploit its fixed locus to eliminate some parameters from q_0 and q_2. After some simplifications, they assume these forms:

$$\begin{aligned} q_0 &= x_1^4 + x_2^4 + x_3^4 + a x_1^2 x_3^2 + a' x_1 x_2^2 x_3 \\ &\quad + y_1 y_3 + b_1 y_1 x_1 x_2 + b_3 y_3 x_2 x_3\,, \\ q_2 &= c_1 x_1^3 x_3 + c_3 x_1 x_3^3 \\ &\quad + d_1 x_1^2 x_2^2 + d_3 x_2^2 x_3^2 + y_1^2 + y_3^2\,. \end{aligned} \tag{4}$$

As in the previous case, we have eight parameters, and we have to eliminate the points which give bad singularities and to quotient by the isomorphisms of underlying Godeaux surfaces.

1.4.1. Theorem. *The coarse moduli space M_4 of Godeaux surfaces with torsion of order 4 is a finite quotient of a nonempty open subset $\tilde{M}_4$ of $\mathbb{A}^8$. A point*

$$(a, a', b_1, b_3, c_1, c_3, d_1, d_3)$$

corresponds to the Godeaux surface obtained resolving the singularities of the quotient by ϱ of the variety defined by equations (4) *in* $\mathbb{P}$.

2.1.5 *Torsion of order three*

Using the same methods as before, we need six generators for the canonical ring of X: x_1, x_2 in degree 1, y_0, y_1, y_2 in degree 2, z_1, z_2 in degree 3. Therefore we should use the tricanonical map to obtain the canonical model of X; actually, we can use just the bicanonical map, since one proves that it is a birational morphism to $\mathbb{P} = \mathbb{P}(1^2, 2^3)$. The image of this morphism is not a complete intersection; it is described by equations:

$$
\begin{aligned}
q_0 &= x_1x_2(y_0^2 - y_1y_2) - x_1^2(y_2^2 - y_0y_1) - x_2^2(y_1^2 - y_0y_2) \\
&\quad + a_1x_1^3x_2y_1 + a_2x_1x_2^3y_2 - b_1x_1^6 + b_{1,2}x_1^3x_2^3 - b_2x_2^6\,, \\
p_0 &= y_0^3 + y_1^3 + y_2^3 - 3y_0y_1y_2 + a_1x_1^2y_0y_1 + a_2x_2^2y_0y_2 \\
&\quad - (a_1 + a_2)x_1x_2y_1y_2 + a_1x_2^2y_1^2 + a_2x_1^2y_2^2 \\
&\quad + (b_1 + b_{1,2} + b_2)x_1^2x_2^2y_0 + b_2x_2^4y_1 - (b_1 + b_{1,2})x_1^3x_2y_1 \\
&\quad + b_1x_1^4y_2 - (b_{1,2} + b_2)x_1x_2^3y_2 + (x_1^3 + x_2^3)S\,, \\
h &= x_1y_1(y_2^2 - y_0y_1) + x_2y_2(y_1^2 - y_0y_2) - a_1x_1^2x_2y_1^2 - a_2x_1x_2^2y_2^2 \\
&\quad - (b_1x_1^3 + b_2x_2^3)x_1x_2y_0 + b_1x_1^5y_1 + b_2x_2^5y_2 - x_1^2x_2^2S\,,
\end{aligned} \tag{5}
$$

where $S = c_1x_1^3 + c_2x_2^3 + d_1x_1y_2 + d_2x_2y_1$.

Actually, omitting h we have the surface $\overline{X}$ plus three fibers of the projective bundle $\mathbb{P} \to \mathbb{P}^1$ (obtained projecting to the first two coordinates), restricted to $(q_0 = 0)$. Moreover, the parameters are not uniquely determined, since they may change by a transformation of the form $x_i \mapsto kx_i$, $y_i \mapsto y_i$, $z_i \mapsto k^{-1}z_i$. Accounting for these transformations, we have the following.

1.5.1. THEOREM. *The coarse moduli space M_3 of Godeaux surfaces with torsion of order 3 is a finite quotient of a nonempty open subset $\tilde{M}_3$ of the weighted projective space $\mathbb{P}(2^2, 4^3, 6^2, 4^2)$. A point*

$$[a_1, a_2, b_1, b_{1,2}, b_2, c_1, c_2, d_1, d_2]$$

corresponds to the Godeaux surface obtained resolving the singularities of the quotient by ϱ of the variety defined by equations (5) in $\mathbb{P}$.

2.2 Automorphism groups of known surfaces

In this section we compute the automorphism groups of all surfaces in the moduli spaces constructed by Reid. The results are split into strata of the moduli spaces and gathered in the three tables 1, 2, and 3. An interesting observation is that the surfaces with the lower torsion amongst the one we consider are all rigid, that is, they do not admit any nontrivial automorphism.

We observe that the way in which the automorphisms are computed reminds of the way one construct a quotient stack. Indeed, we prove that the moduli stack of numerical Godeaux surfaces with torsion group of order five is a quotient stack. We do this using the automorphisms computation, so we are able to describe explicitly the structure of this stack.

2.2.1 *Computing automorphisms*

This section is the technical heart of the chapter: in the first subsection we will discuss the mathematics needed to solve the problem; in the second, as an example, we will apply it to the case of torsion of order $\nu = 5$, without doing any hard computation; in the third we will explain the structure of the program that does the computations, referring to the example for clarifications.

Here we let S be again a Godeaux surface with torsion isomorphic to $\mathbb{Z}_\nu$ with $\nu \geq 3$, and $\psi\colon X \to S$ its universal Galois cover. Moreover we take $\varphi\colon X \to \overline{X} \subset \mathbb{P}$ to be the canonical (in the case of $\nu = 5$) or bicanonical (in the other cases) birational morphism as constructed in the previous section.

2.2.1.1 The strategy. We will use without further mention the following facts.

(1) An automorphism of $X \subset \mathbb{P}$ extends to an automorphism of $\mathbb{P}$ (in particular is described by a matrix in $\mathbb{P}\,\mathrm{GL}(n+1)$).
(2) An isomorphism of two Godeaux surfaces S_1 and S_2 lifts to an automorphism of the universal covers X_1 and X_2.
(3) The automorphisms of a universal cover X that pass to the quotient to automorphisms of S are the ones compatible with the action of G, i.e. the ones in the normalizer of $\mathrm{Aut}(X)$ relative to the action of G. The kernel of the map $\mathrm{N}_{\mathrm{Aut}(X)}(G) \to \mathrm{Aut}(S) \to 0$ is simply G.

In every case we studied before, we fixed the action of the torsion group G on the projective space $\mathbb{P}$; hence the compatibility with the action does not depend on the particular equations of $\overline{X}$. Up to now we can describe $\mathrm{Aut}(S)$ as the quotient by G of $\mathrm{Aut}(X)$, and we can represent elements of $\mathrm{Aut}(X)$ by matrices in $\mathbb{P}\,\mathrm{GL}(n+1)$. Firstly, we have to reduce the possibilities for these matrices.

2.1.1. LEMMA. *A matrix A representing an automorphism α of X is a permutation of a diagonal matrix. In particular, the permutation is induced by the action of α on the eigenspaces of* $\mathrm{H}^0(nK_X)$ *relative to G.*

PROOF. The automorphism α induces an automorphism $\alpha^\star$ of $\mathrm{H}^0(X, nK_X)$ for every $n \geq 0$. This gives as a result that A is a block matrix (i.e. $\alpha^\star(\bullet_i) = \bullet_j$ for some j, where $\bullet$ represents the letter x or y, when it makes sense). Since α is by definition compatible with the action of G, also $\alpha^\star$ is compatible with the action of G on $\mathrm{H}^0(X, nK_X)$, so there is an induced permutation of the eigenspaces relative to G, i.e. $\alpha^\star$ acts on the characters of G as an element of $\mathrm{Aut}(G) \cong G^\star$.

It is now easy to see that A has to be a permutation of a diagonal matrix; indeed, if α is in the class relative to $g \in \mathbb{Z}_\nu^\star$, then $\alpha^\star$ sends $\bullet_i$ to $\bullet_{gi}$ and this is well defined since in our cases for every n and every g the eigenspace of $\mathrm{H}^0(X, nK_X)$ with eigenvalue g is at most 1-dimensional. □

So, if $\nu = 5$ the possible automorphisms are divided in four classes (1, 2, 3, and 4); if $\nu = 4$ there are two classes (1 and 3); if $\nu = 3$ there are two classes (1 and 2). Moreover, for every class we have a fixed permutation and these permutations form a group isomorphic to $\mathbb{Z}_\nu^\star$. Hence, the automorphism groups we will find will be semidirect products of some finite group by a subgroup of $\mathbb{Z}_\nu^\star$.

Let us denote with γ_ν the number of generators of the canonical ring considered in the previous section, or equivalently the dimension of the projective space in which $\overline{X}$ is embedded. Up to now, to describe an automorphism of X we need $\gamma_\nu - 1$ complex parameters (because we work in $\mathrm{Aut}(\mathbb{P})$). In the following Lemma, we show that these parameters cannot be generic.

2.1.2. LEMMA. *Up to normalizing, the nonzero entries of A are ν-th roots of unity.*

PROOF. We define λ_i and μ_i implicitly by $\alpha^\star x_i = \lambda_i x_{gi}$ and $\alpha^\star y_i = \mu_i y_{gi}$.

CASE $\nu = 5$. In the equation q_0 defining X we have the terms x_i^5; hence, to send q_0 in a multiple of itself, the ratios of the four parameters must be fifth roots of unity; if we normalize one of them to 1, the others must be of the form ξ^{i_g}.

CASE $\nu = 4$ If we normalize λ_2 to 1, we have from q_0 that λ_1, λ_3, and $\mu_1\mu_3$ are fourth roots of unity; from q_2 we have that $\mu_1^2 = \mu_3^2$, hence also μ_1 and μ_3 are fourth roots of unity. Moreover we also observe that $\mu_3 = \mu_1^{-1}$.

CASE $\nu = 3$. We can normalize λ_1 to 1, so that from p_0 we have $\mu_1 = \mu\xi^{j_1}$, $\mu_2 = \mu\xi^{j_2}$ and $\mu_0 = \mu\xi^{2j_1+2j_2}$, and from q_0 we have

$$\lambda_2\mu^2\xi^{j_1+j_2} = \mu^2\xi^{2j_2} = \lambda_2^2\mu\xi^{2j_1};$$

from these equations, we get $\lambda_2 = \xi^{2j_1+j_2}$. We still have a continuous parameter; to kill it, we have to exploit the fact that $a_1 + a_2 \neq 0$ (see from **[Rei78]**): this allows us to say that $\mu_0^3 = \lambda_2\mu_1\mu_2$, i.e. $\mu^3 = \xi^{2j_2}$. □

Let P_σ be the matrix associated to the permutation σ, corresponding to the multiplication by an element of $\mathbb{Z}_\nu^\star$. Lemma 2.1.2 tells us that the possible matrices A representing an automorphism α of X are of the following forms.

CASE $\nu = 5$. $A = \operatorname{diag}(1, \xi^{i_2}, \xi^{i_3}, \xi^{i_4})P_\sigma$.

CASE $\nu = 4$. $A = \operatorname{diag}(\xi^{i_1}, 1, \xi^{i_3}, \xi^{j_1}, \xi^{-j_1})P_\sigma$.

CASE $\nu = 3$. Taking $k = 2j_1 + j_2$, $A = \operatorname{diag}(1, \xi^k, \xi^k, \xi^{2k}, 1)P_\sigma$.

2.1.3. REMARK. Up to now, we did not use that $\alpha(X) = X$; that is, these matrices represent all the isomorphisms between points of $\tilde{M}_\nu$. Let $H_\nu \subset \operatorname{Aut}(\mathbb{P})$ be the group consisting of all these matrices; then $H_\nu = \mathrm{N}_{\operatorname{Aut}(\mathbb{P})}(G)$ and $\tilde{M}_\nu/H_\nu$ is the coarse moduli space M_ν of Godeaux surfaces with torsion of order ν. In particular, it is $\mathbb{Z}_5^3 \ltimes \mathbb{Z}_5^\star$ for $\nu = 5$, $\mathbb{Z}_4^3 \ltimes \mathbb{Z}_4^\star$ for $\nu = 4$ and the symmetric group of order 6, $\mathbb{Z}_3 \ltimes \mathbb{Z}_3^\star$, for $\nu = 3$. Note that the torsion group $G \subset H_\nu$ acts trivially on $\tilde{M}_\nu$.

Coming back to automorphisms, we have proved that for a given X, there is only a finite group of possible automorphisms. Depending on the actual equations defining X, $\operatorname{Aut}(X)$ is a subgroup of that group. In particular, we experience changing of $\operatorname{Aut}(X)$ when some parameters becomes zero or when the ratios of two parameters related by a permutation becomes a nu-th root of unity. If we work on the parameters' affine space instead that on the coarse moduli space, i.e. on $\tilde{M}_\nu = \mathbb{A}^8$ for $\nu \in \{5, 4\}$, and on $\tilde{M}_\nu = \mathbb{A}^9$ for $\nu = 3$, these changes of $\operatorname{Aut}(X)$ happens only in some vector subspaces of $\tilde{M}_\nu$. Even with this simplification, there are eight or nine parameters which gives hundreds of different cases; this is the reason to use a program to automate the computations.

2.2.1.2 Example: the case $\nu = 5$. To explain the program, we present the easier case. When $\nu = 5$, we have four generators,

x_1, x_2, x_3 and x_4, with degree 1, with only one relation in degree 5. An automorphism α of X is represented by a matrix of the form $\mathrm{diag}(1, \xi^{i_2}, \xi^{i_2}, \xi^{i_3})P_\sigma$, where σ is the permutation given by the multiplication by an element of $\mathbb{Z}_5^\star$. We have to compute the possible automorphisms, one permutation a time.

If σ is the identity, then the generic equation (3) is transformed by α to

$$\begin{aligned}\alpha^\star q_0 = {} & x_1^5 + x_2^5 + x_3^5 + x_4^5 \\ & + b_1\xi^{i_2+3i_3+i_4}x_2x_3^3x_4 + b_2\xi^{i_3+i_4}x_1^3x_3x_4 \\ & + b_3\xi^{i_2+3i_4}x_1x_2x_4^3 + b_4\xi^{3i_2+i_3}x_1x_2^3x_3 \\ & + c_1\xi^{2i_2+i_3+2i_4}x_2^2x_3x_4^2 + c_2\xi^{2i_3+2i_4}x_1x_3^2x_4^2 \\ & + c_3\xi^{2i_2+i_4}x_1^2x_2^2x_4 + c_4\xi^{i_2+2i_3}x_1^2x_2x_3^2\,.\end{aligned}$$

Since the terms x_s^5 are unchanged, the condition on i_2, i_3 and i_4 for α to fix X is $\alpha^\star q_0 = q_0$, i.e. this system of equations in $\mathbb{Z}_5$:

$$\left\{\begin{array}{ll} i_2 + 3i_3 + i_4 \equiv 0, & 2i_2 + i_3 + 2i_4 \equiv 0, \\ i_3 + i_4 \equiv 0, & 2i_3 + 2i_4 \equiv 0, \\ i_2 + 3i_4 \equiv 0, & 2i_2 + i_4 \equiv 0, \\ 3i_2 + i_3 \equiv 0, & i_2 + 2i_3 \equiv 0\,, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} i_3 \equiv 2i_2, \\ i_4 \equiv 3i_2\,. \end{array}\right.$$

Obviously this is so if all b_s and c_s are nonzero; if some of them are zero, then the associated equations are not in the system.

If σ is not the identity, there are also some swaps amongst the coefficients b_s and c_s; for example if σ corresponds to the multiplication by 4, we have

$$\begin{aligned}\alpha^\star q_0 = {} & x_1^5 + x_2^5 + x_3^5 + x_4^5 \\ & + b_4\xi^{3i_2+i_3}x_2x_3^3x_4 + b_3\xi^{i_2+3i_4}x_1^3x_3x_4 \\ & + b_2\xi^{i_3+i_4}x_1x_2x_4^3 + b_1\xi^{i_2+3i_3+i_4}x_1x_2^3x_3 \\ & + c_4\xi^{i_2+2i_3}x_2^2x_3x_4^2 + c_3\xi^{2i_2+i_4}x_1x_3^2x_4^2 \\ & + c_2\xi^{2i_3+2i_4}x_1^2x_2^2x_4 + c_1\xi^{2i_2+i_3+2i_4}x_1^2x_2x_3^2\,,\end{aligned}$$

and for α to fix X we need again $\alpha^\star q_0 = q_0$. But because of the nontrivial permutation, necessary conditions to have an automorphisms are that b_1/b_4 is a fifth root of unity and the same for all the coefficients swapped. We define $n_{s,t}$ and $m_{s,t}$ in such a way that $b_s/b_t = \xi^{n_{s,t}}$ and $c_s/c_t = \xi^{m_{s,t}}$, assuming this is possible. This time the system of equations will give conditions not only on the entries of α, but also on the coefficients of q_0:

$$\begin{cases} n_{1,4} \equiv 3i_2 + i_3, & m_{1,4} \equiv i_2 + 2i_3, \\ -n_{3,2} \equiv i_2 + 3i_4, & -m_{3,2} \equiv 2i_2 + i_4, \\ n_{3,2} \equiv i_3 + i_4, & m_{3,2} \equiv 2i_3 + 2i_4, \\ -n_{1,4} \equiv i_2 + 3i_3 + i_4, & -m_{1,4} \equiv 2i_2 + i_3 + 2i_4\,, \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} n_{3,2} \equiv 2n_{1,4}, \\ m_{1,4} \equiv 2n_{1,4}, \\ m_{3,2} \equiv 4n_{1,4} \end{cases} \wedge \quad \begin{cases} i_3 \equiv 2i_2 + n_{1,4}, \\ i_4 \equiv 3i_2 + n_{1,4} \end{cases}$$

This means that even if all coefficients are nonzero, we will have automorphisms with this permutation only in the five four-dimensional subspaces of $\tilde{M}_5$ defined by

$$b_1 = b_4 \zeta^{n_{1,4}}\,, \qquad b_3 = b_2 \zeta^{2n_{1,4}}\,,$$
$$c_1 = c_4 \zeta^{2n_{1,4}}\,, \qquad c_3 = c_2 \zeta^{4n_{1,4}}\,.$$

Continuing with the last two permutations, doing all the computations, and combining all the data collected, we arrive at the complete description of the automorphism group of Godeaux surfaces with torsion of order 5.

2.2.1.3 **Description of the program.** In this section we will describe the program we wrote to compute the automorphism groups. It is available at [**Mag**]. It is written in *Python*, using the library *sympy* to handle symbolic computation. It also uses *GAP*, mainly to identify the groups we obtain at the end of the computation.

Here are the main classes, with their methods.

(1) The class `GAPInterface` connects the main program with *GAP*. Its public methods are:
   - `NullSpaceMat`, which returns the kernel of a matrix given as input;
   - `IdSmallGroup`, which returns the id of a group in the *GAP*'s small group list; the group is passed as a list of generators and a list of relations.

   It uses internally `rewrite_expr`, which translate an expression from *sympy* to *GAP*.
(2) The class `LinearModularParametricSystem` solves a linear system in the ring $\mathbb{Z}_\nu$, $3 \le \nu \le 5$; it is parametric in the sense that some unknowns are treated as parameters, and, in the solution, the the value of a parameter cannot depend on a regular unknown. Its main methods are:
   - `solve`, with the obvious meaning;

- `iter_solutions`, returning an iterator through all the possible values of the regular unknowns (eventually depending on the parameters);
- `gens_sample_solutions`, same as before, but substituting a sample values for the parameters (i.e. all zeroes) and returning only the generators of the solutions;
- `iter_pars_solutions`, returning an iterator through all the possible values of the parameters.

(3) The class `VectorSpace` implements complex vector subspaces: it takes as input two lists, of generators and of linear equations.
(4) The class `GodeauxAutomorphismComputer` is where the actual computation is done. We will describe it in detail later.

We have three functions which define the needed data for the three cases and call `GodeauxAutomorphismComputer`. The input data are the following (between parenthesis the data for the example $\nu = 5$):

(1) `n` (5), the order of the torsion group;
(2) `monomials` ($x_1^5, \ldots, x_1^2 x_2 x_3^2$), the monomials involved in the equations of X;
(3) `mod_pars` ($b_1, \ldots, b_4, c_1, \ldots c_4$), the basis of $\tilde{M}_\nu$;
(4) `cr_gens` ($x_1, \ldots, x_4$), the generator of the cohomology spaces $\mathrm{H}^0(X, K_X)$ or $\mathrm{H}^0(X, 2K_X)$; in the latter case, if the generators were x_s in degree 1 and y_s in degree 2, we put the y_s and the products $x_s x_t$ denoted as $x_{s,t}$;
(5) `cr_rels` (equation (3)), the relations between elements of `cr_gens`, depending on `mod_pars`, excluding the trivial ones such as $x_{s,t} x_{u,v} = x_{s,u} x_{t,v}$;
(6) `cr_rels_multiplier` (1), the coefficients by which the relations `cr_rels` get multiplied after applying an automorphism; since in every equation there is a constant monomial, we know this coefficient;
(7) `sys_unks` (i_2, i_3, i_4), the list of unknown exponents of ξ in the definition of the general automorphism α;
(8) `sys_pars` ($n_{i,j}, m_{i,j}$), the list of possible parameters showing up in the computations;
(9) `sys_pars_coupling` ($(b_i, b_j) \mapsto n_{i,j}, \ldots$), a dictionary that associate a parameter in `sys_pars` to a ratio between two coefficients in `mod_pars`;
(10) `alpha` ($\mathrm{diag}(1, \xi^{i_2}, \xi^{i_3}, \xi^{i_4})$), the generic automorphism with $\sigma = 1$;

(11) `perms` ($I, P = P_{(2,4,3,1)}, P^2, P^3$), a dictionary that associate to a number in $\mathbb{Z}_\nu^\star$ the permutation matrix; in particular we get the generic automorphism with permutation h as `alpha * perms[h]`;
(12) `rho` ($\mathrm{diag}(1, \xi, \xi^2, \xi^3)$), the matrix representing a generator for the action of $\mathbb{Z}_\nu$ on $\mathbb{P}$;
(13) `psi` ($\mathrm{diag}(\xi, \xi, \xi, \xi)$), generators for the group to quotient by to obtain $\mathbb{P}\,\mathrm{GL}(n+1)$ from $\mathrm{GL}(n+1)$; this is needed since *GAP* does not understand projective matrices groups.

The class `GodeauxAutomorphismComputer` splits the computation in three steps, each of which consisting in a private method.

(1) The first method, `compute_equations`, builds the dictionary `equations`, indexed by permutations and pairs of elements of `mod_pars`, of modular equations that will compose the systems to be solved. For example, if $\nu = 5$, the entry corresponding to the permutation 4 and parameters (b_4, b_1) is the equation $-n_{1,4} \equiv i_2 + 3i_3 + i_4$. The dictionary is built applying the generic automorphism `alpha * perms[h]` and comparing the coefficients of the elements of `monomials`.
(2) The second method, `compute_solutions`, iterates through all possible vanishing of elements in `mod_pars`, that is in $\{0,1\}^8$ or $\{0,1\}^9$; for every vanishing and every permutation, it takes the equations from `equation` and summon `LinearModularParametricSystem` to solve it. After this, it computes the relations between the parameters needed to have solutions, that is, the vector subspace where the solutions live. It builds a dictionary, `automorphisms_gens`, indexed by the various vector spaces and with values the set of matrices generating the automorphism group found solving the system. The last thing it does is to propagate the set of automorphisms of a larger vector space V to the set of vector space contained in V.
(3) The third method, `regroup_solutions`, takes all this information, spread in all the vector spaces and collects them together. Firstly, it computes *GAP*'s id for all the possible set of generators, and build a dictionary indexed by these ids and with values the list of vector spaces which have that group as automorphism group. Then it remove from these lists irrelevant vector spaces, that is the ones that are contained in a different space with the same automorphism group.

Group	*GAP* id	V. sp.	Equations		D	C
$\{1\}$	$(1,1)$	$\tilde{M}_5$			8	1
$\mathbb{Z}_2$	$(2,1)$	Q_s	$b_1 = b_4\zeta^s$ $b_3 = b_2\zeta^{2s}$	$c_1 = c_4\zeta^{2s}$ $c_3 = c_2\zeta^{4s}$	4	5
$\mathbb{Z}_4$	$(4,1)$	$P_{s,t}$	$b_1 = b_2\zeta^{s+t}$ $b_2 = b_4\zeta^s$ $b_3 = b_1\zeta^{3s+t}$	$c_1 = c_2\zeta^{2s+2t}$ $c_2 = c_4\zeta^{2s}$ $c_3 = c_1\zeta^{s+2t}$	2	25
$\mathbb{Z}_5$	$(5,1)$	H_u	$b_v = c_v = 0,\ \forall v \neq u$		2	4
$\mathbb{Z}_5^2 \ltimes \mathbb{Z}_4$	$(100,10)$	O	$b_v = c_v = 0,\ \forall v$		0	1

TABLE 1. Special subcomponents in the case $\nu = 5$ (D is the dimension and C the number of components).

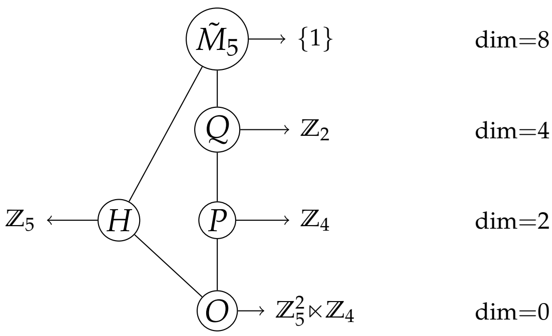


FIGURE 1. Hasse diagram for $\nu = 5$.

2.2.2 *The results*

2.2.2.1 Torsion of order five. The results given by the program are listed in Table 1 (where $s, t \in \mathbb{Z}_5$ and $u, v \in \mathbb{Z}_5^\star$).

We also have the relations of containment amongst the various vector spaces, recorded in Figure 1 (a vertical path means that the space at the lower end is contained in the one at the upper end).

2.2.1. REMARK. We worked in $\tilde{M}_5$; it may happen that some of them lie in the locus of $\tilde{M}_5$ we have to wipe out because of bad singularities. This is not the case for $\nu = 5$: we know that the origin O represents a Godeaux surface (actually, the one Godeaux himself constructed). Hence, the space of Godeaux surfaces is a nonempty open set in every subspace we consider, since each one passes through the origin.

It is easy to see that the high number of components in the three middle cases are due to the fact that up to now we are considering $\tilde{M}_5$ and not M_5 itself. Indeed, passing to the quotient, all the components collapse in M_5 to one irreducible component for each group.

2.2.2.2 Torsion of order four. The results for the case $\nu = 4$ are given in Table 2 (where $s, v \in \mathbb{Z}_4^\star$). In this case, the origin does not represent anymore a Godeaux surface. Indeed, in the origin we have $q_2 = y_1^2 + y_3^2$ which is reducible. Therefore, the argument of Remark 2.2.1 does not apply. We will show later that the three vector spaces with a bullet on the right are exactly the ones not containing Godeaux surfaces.

Again, when a space has several components, they collapse to one in M_4; moreover, we can easily check that R_2 and R_3 collapse into one irreducible component inside M_4; the same is true for S_2 and S_3, and T_2 and T_3. We define $R_{2,3} = R_2 \cup R_3$, $S_{2,3} = S_2 \cup S_3$ and $T_{2,3} = T_2 \cup T_3$.

As we did before, we represent all the vector spaces into Figure 2, ordered by containment. We recall that a vertical path means containment, but here we have also vertical dashed segments: for example, the one connecting $T_{4,s}$ with $S_{6,s}$ means that the former is not contained in the latter, but it is so when seen in the quotient M_4. Also, vector spaces with dashed circle are the same as marked vector spaces in the table (that is, they do not contain any point representing Godeaux surfaces). To prove that they are exactly the spaces not containing Godeaux surfaces, we "climb" the diagram starting from the origin O, and for each space we check if it contains some (equivalently, an open subset of) Godeaux surfaces.

We already seen that O cannot corresponds to a Godeaux surface, since $q_2 = y_1^2 + y_3^2$ is reducible. For the same reason, T_1 and S_1 do not contain points corresponding to Godeaux surfaces.

In T_2 we have these equations:

$$q_0 = x_1^4 + x_2^4 + x_3^4 + a x_1^2 x_3^2 + y_1 y_3 \,,$$
$$q_2 = d_1 x_1^2 x_2^2 + y_1^2 + y_3^2 \,;$$

dehomogenizing by x_i, $i \in \{1, 2, 3\}$, we obtain an affine covering of X, so we can compute the singularities via the Jacobian matrix. For example, in the open set $(x_1 \neq 0) \cong \mathbb{A}^4$ we have

$$J = \begin{pmatrix} 4x_2^3 & 4x_3^3 & y_3 & y_1 \\ 2d_1 x_2 & 0 & 2y_1 & 2y_3 \end{pmatrix} \,;$$

thanks to the last minor involving only the y_i, we have two cases in which J has rank strictly less than 2.

Group	*GAP* id	V. sp.	Equations	D	C	
$\{1\}$	$(1,1)$	$\tilde{M}_4$		8	1	
$\mathbb{Z}_2$	$(2,1)$	R_1	$b_v=0,\ \forall v$	6	1	
		W_s	$b_1=b_3\xi^s$ $c_1=c_3\xi^{2s}\quad d_1=d_3\xi^{2s}$	5	2	
		R_2	$a'=b_3=c_v=0,\ \forall v$	4	1	
		R_3	$a'=b_1=c_v=0,\ \forall v$	4	1	
$\mathbb{Z}_2^2$	$(4,2)$	$R_{4,s}$	$b_v=0,\ \forall v$ $c_1=c_3\xi^s\quad d_1=d_3\xi^s$	4	2	
		$R_{5,s}$	$a'=b_1=b_3=0$ $c_1=c_3\xi^s\quad d_1=d_3\xi^{s+2}$	3	2	
		S_5	$a'=b_v=c_v=0,\ \forall v$	3	1	
$\mathbb{Z}_4$	$(4,1)$	S_4	$a'=b_v=d_v=0,\ \forall v$	3	1	
		S_2	$a=a'=b_3=c_v=d_3=0,\ \forall v$	2	1	
		S_3	$a=a'=b_1=c_v=d_1=0,\ \forall v$	2	1	
$\mathbb{Z}_4\times\mathbb{Z}_2$	$(8,2)$	$S_{6,s}$	$a'=b_v=d_v=0,\ \forall v$ $c_1=c_3\xi^s$	2	2	
		$T_{4,s}$	$a=a'=b_v=d_v=0,\ \forall v$ $c_1=c_3\xi^{s+1}$	1	2	
		T_2	$a=a'=b_v=c_v=d_3=0,\ \forall v$	1	1	
		T_3	$a=a'=b_v=c_v=d_1=0,\ \forall v$	1	1	
D_8	$(8,3)$	S_1	$b_v=c_v=d_v=0,\ \forall v$	2	1	•
		$S_{7,s}$	$a'=b_v=c_v=0,\ \forall v$ $d_1=d_3\xi^s$	2	2	
$G_{(16,13)}$	$(16,13)$	T_1	$a'=b_v=c_v=d_v=0,\ \forall v$	1	1	•
$\mathbb{Z}_4^2\ltimes\mathbb{Z}_2$	$(32,11)$	O	$a=a'=b_v=c_v=d_v=0,\ \forall v$	0	1	•

TABLE 2. Special subcomponents in the case $\nu=4$ (D is the dimension and C the number of components). The group $G_{(16,13)}$ if $(\mathbb{Z}_4\times\mathbb{Z}_2)\ltimes\mathbb{Z}_2$.

(1) If $y_1=y_3=0$, the other minors, involving only the x_i, have to be 0, so we have the following equations:

$$0=8d_1x_2x_3^3=1+x_2^4+x_3^4=d_1x_2^2\,.$$

Since we are interested in an open subset of T_2, we may assume $d_1\neq 0$ so we get four singular points $x_2=0$, $x_3=\xi^j\sqrt[4]{-1}$.

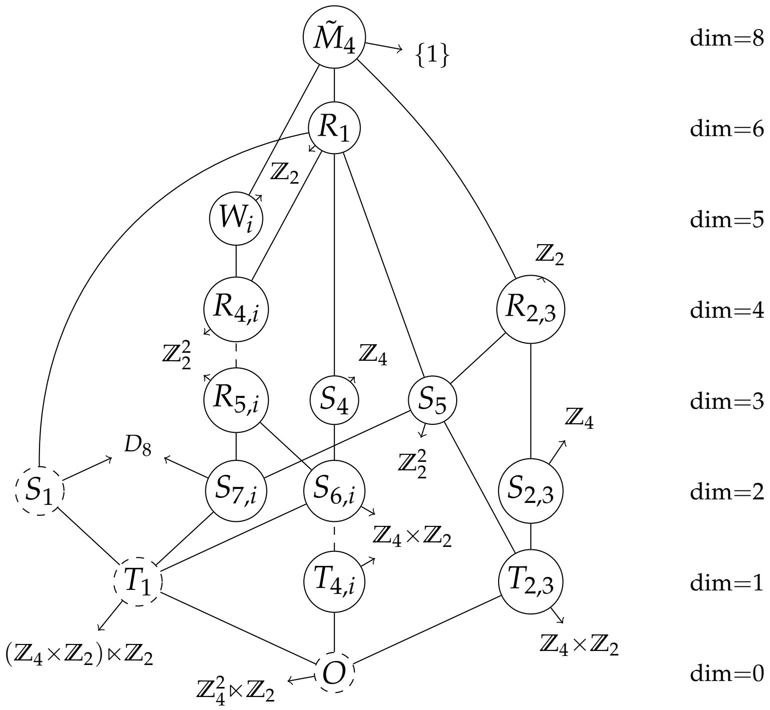


FIGURE 2. Hasse diagram for $\nu = 4$.

(2) If $y_3 = \pm y_1 \neq 0$, then the second row is plus or minus two times the first row; in particular we have

$$0 = x_3 = 8x_2^3 \mp 2d_1x_2 = 1 + x_2^4 \pm y_1^2 = d_1x_2^2 + 2y_1^2 .$$

It cannot happen that $x_2 = 0$, so $x_2^2 = d_1/4$ and $y_1^2 = -d_1^2/8$; but this implies $d_1^2 = 16$ and we can discard this particular situation that happens only in a proper closed subset of T_2.

In the same way we can find singular points in the other two affine open subsets $(x_2 \neq 0)$ and $(x_3 \neq 0)$, and the result is that we have 8 singular points for the surface X represented by a generic point of T_2:

$$[1, 0, \xi^j \sqrt[4]{-1}, 0, 0] , \qquad\qquad [0, 1, \xi^j \sqrt[4]{-1}, 0, 0] .$$

Now we have to check if these are rational double points or worse. For example, consider the point $p = [1, 0, \xi^j \sqrt[4]{-1}, 0, 0]$ in the affine open set relative to x_1, we have $(\partial q_0/\partial x_3)|_p \neq 0$, hence we can represent, analytically locally, x_3 as $x_3(p) + g(x_2, y_1, y_3)$. Substituting x_3 in q_2, we obtain the expression

$$q_2 = y_1^2 + y_3^2 + x_3(p)d_1x_2^2 + \cdots$$

where $x_3(p) \neq 0$ and the other terms are of order at least three in p. So the singularity is of type A_1, in particular it is a rational double

Group	*GAP* id	V. sp.	Equations	D	C	
$\{1\}$	$(1,1)$	$\tilde{M}_3$		9	1	
$\mathbb{Z}_2$	$(2,1)$	A	$a_1 = a_2 = b_1 = b_2 = 0$	5	1	•

TABLE 3. Special subcomponents in the case $\nu = 3$ (D is the dimension and C the number of components).

point. The situation is the same for every other singular point (since they're in the same G-orbit), so we conclude that in T_2 there is a nonempty open set of Godeaux surfaces.

The situation in T_3 is completely specular. We do not write the similar computation for $T_{4,i}$ and $S_{7,i}$ anyway both contain an open subset of Godeaux surfaces.

2.2.2.3 **Torsion of order three.** The results given from the program in the case $\nu = 3$ are simpler then the others. This is understandable: going from $\mathbb{Z}_5$ to $\mathbb{Z}_4$ we've seen an increasing complexity on the vector spaces, but a decreasing order of the automorphism groups. In this last case, the latter behaviour prevails on the former.

Indeed, the results listed in Table 3 are just two lines, the second of them describing a vector space not containing any Godeaux surface (we already use that for a point to describe a Godeaux surface, it must be $a_1 + a_2 \neq 0$); i.e. Godeaux surfaces with torsion of order three have no nontrivial automorphisms.

2.2.3 *Moduli stacks*

In this section we define the moduli stack $\mathfrak{M}_\nu$ of Godeaux surfaces with torsion of order ν and relate it to the computation of automorphisms of the previous section. More precisely, let $G \cong \mathbb{Z}_\nu$ be the torsion group of a Godeaux surface S realized as a subgroup of $\mathrm{Aut}(\mathbb{P})$ as in 1.3, $H_\nu = \mathrm{N}_{\mathrm{Aut}(\mathbb{P})}(G)$ as in Remark 2.1.3, and denote with $\mathcal{M}_\nu$ the quotient stack $[\tilde{M}_\nu/(H_\nu/G)]$. We will show that there is a natural map $\Phi\colon \mathcal{M}_\nu \to \mathfrak{M}_\nu$ and that it is an equivalence on points. Moreover, we will show that this map is an isomorphism in the case $\nu = 5$; there are no reasons to doubt that this holds also for the other torsions. Nevertheless we would need finer arguments, since the description of the canonical model of a surface with lower torsion is not as nice as in the case $\nu = 5$.

2.3.1. REMARK. There are two natural definitions for the moduli stack of surfaces: the first considers flat projective families where the fibers are smooth minimal models of some surface in the class; the second considers canonical models instead of minimal models. Often the latter seems more natural than the former and here we will pursue this approach. Recall that for a Godeaux surface S, we denoted with $X \to S$ the smooth cover coming from $\mathrm{Tors}(S)$, and with $\overline{X}$ and $\overline{S}$ the canonical models of X and S.

2.3.2. DEFINITION. The *moduli stack of Godeaux surfaces* with torsion of order ν is the stack $\mathfrak{M}_\nu$ defined as a category fibered in groupoids by:

$$\mathrm{Obj}(\mathfrak{M}_\nu) = \left\{ \pi\colon \overline{S}_B \to B \;\middle|\; \begin{array}{l} \pi \text{ flat, projective,} \\ \forall b \in B,\ \overline{S}_b \text{ is the canonical} \\ \text{model of a Godeaux surface} \\ \text{with torsion of order } \nu \end{array} \right\}, \text{ and}$$

$$\mathrm{Mor}_{\mathfrak{M}_\nu}(\pi, \pi') = \left\{ (\varphi, \psi) \;\middle|\; \begin{tikzcd} \overline{S}_B \arrow[r, "\psi"'] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \overline{S}'_{B'} \arrow[d, "\pi'"'] \\ B \arrow[r, "\varphi"] & B' \end{tikzcd} \right\};$$

the projection to schemes sends $\pi\colon \overline{S}_B \to B$ to B and (φ, ψ) to φ.

2.3.3. PROPOSITION. *There is a natural morphism of stacks $\Phi\colon \mathcal{M}_\nu \to \mathfrak{M}_\nu$.*

PROOF. Let $\tilde{\Phi}\colon \tilde{M}_\nu \to \mathfrak{M}_\nu$ be the morphism determined by the universal family $U \to \tilde{M}_\nu$, with $U \subseteq \tilde{M}_\nu \times (\mathbb{P}/G)$. We will prove that $\tilde{\Phi}$ is H_ν-equivariant, and so it passes to the quotient (recall that $G \subset H_\nu$ acts trivially on $\tilde{M}_\nu$).

Being H_ν-equivariant means that for every $h \in H_\nu$ we have a canonical 2-morphism η making the following diagram 2-commutative:

$$\begin{tikzcd} \tilde{M}_\nu \arrow[rr, "h"] \arrow[dr, "\tilde{\Phi}"'] \arrow[rr, phantom, shift right=4, "\Nearrow \eta"] & & \tilde{M}_\nu \arrow[dl, "\tilde{\Phi}"] \\ & \mathfrak{M}_\nu . & \end{tikzcd}$$

Given a map $f\colon T \to \tilde{M}_\nu$, we have that $\tilde{\Phi}(f)$ is the family $\overline{S}_T \to T$ in the cartesian diagram

$$\begin{tikzcd} \overline{S}_T \arrow[r] \arrow[d, "\tilde{\Phi}(f)"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "u"] \\ T \arrow[r, "f"'] & \tilde{M}_\nu \end{tikzcd}$$

and in the same way $\tilde{\Phi} \circ h(f)$ is the family $\overline{S}'_T \to T$. We have to define $\eta(f)\colon \tilde{\Phi}(f) \Rightarrow \tilde{\Phi} \circ h(f)$ as a couple of morphisms $(g, \bar{g})$ making the following diagram cartesian:

$$\begin{tikzcd} \overline{S}_T \arrow[r, "\bar{g}"] \arrow[d, "\tilde{\Phi}(f)"'] & \overline{S}'_T \arrow[d, "\tilde{\Phi}(h(f))"] \\ T \arrow[r, "g"'] & T \end{tikzcd}$$

Since

$$\overline{S}_T = T \times_{\tilde{M}_\nu} U \subseteq T \times_{\tilde{M}_\nu} (\tilde{M}_\nu \times (\mathbb{P}/G)) \cong T \times (\mathbb{P}/G)\,,$$

over every point $t \in T$ we have the natural isomorphism $h\colon \overline{S}_{T,t} \to \overline{S}'_{T,t}$, that extends to $\psi\colon \overline{S}_T \to \overline{S}'_T$, and we define $\eta(f) = (\mathrm{id}_T, \psi)$. $\square$

2.3.4. LEMMA. *The morphism Φ induces an equivalence of groupoids $\Phi(\mathbb{C})\colon \mathcal{M}_\nu(\mathbb{C}) \to \mathfrak{M}_\nu(\mathbb{C})$.*

PROOF. An object of $\mathcal{M}_\nu(\mathbb{C})$ is a diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\pi"'] & \tilde{M}_\nu \\ \operatorname{Spec} \mathbb{C} & \end{tikzcd}$$

with π an (H_ν/G)-torsor and f an (H_ν/G)-equivariant morphism; in other words,

$$\mathcal{M}_\nu(\mathbb{C}) = \left\{ (T, f) \;\middle|\; \begin{array}{l} T \cong H_\nu/G \text{ as schemes} \\ H_\nu/G \text{ acts freely on } T \\ f \ (H_\nu/G)\text{-equivariant} \end{array} \right\}$$

As a consequence, all the points of T are mapped to points of $\tilde{M}_\nu$ corresponding to the same Godeaux surface, modulo isomorphism, that is the image of the object via $\Phi(\mathbb{C})$.

We will prove that $\Phi(\mathbb{C})$ is essentially surjective and that is bijective on morphisms.

ESSENTIALLY SURJECTIVE. The translation of this is that for every Godeaux surface S there exists a point in $\mathcal{M}_\nu(\mathbb{C})$ sent to a surface isomorphic to the canonical model of S. The object $(H_\nu/G, f)$ will do if $f(e) \in \tilde{M}_\nu$ is a point corresponding to S, and f is extended equivariantly.

BIJECTION ON MORPHISMS. For this, we have to prove that the automorphisms of (T, f) are in a bijection with the automorphisms of $\overline{S} = \Phi(\mathbb{C})(T, f)$; but, this is exactly what we proved in the previous section, since automorphisms of (T, f) are in a bijection with stabilizers of H_ν/G over a point $f(t)$ for $t \in T$ (this does not change when t changes since all $f(t)$ are in the same orbit). $\square$

We recall here a useful statement.

2.3.5. LEMMA. *Let X and Y be smooth stacks of dimension d; then a morphism $f\colon X \to Y$ is an isomorphism if and only if*

(1) $f(\operatorname{Spec}\mathbb{C})$ *is an equivalence of groupoids, and*
(2) f *is bijective on tangent vectors.*

2.3.6. REMARK. For a Godeaux surface S, Riemann-Roch yields

$$\chi(\mathrm{T}_S) = 2K_S^2 - 10\chi(\mathcal{O}_S) = -8\,,$$

hence $\mathrm{h}^2(S, \mathrm{T}_S) = 0$ if and only if $\mathrm{h}^1(S, \mathrm{T}_S) = 8$. If S is a Godeaux surface with singular canonical model, we can still define the Euler characteristic of the pair $(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}})$ to be

$$\chi(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = \sum_{i=0}^{2} \operatorname{ext}^i(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}})\,,$$

generalizing the previous one. We know that S can be deformed to a Godeaux surface S' with smooth canonical model; since the dimensions of the Ext groups are deformation invariants, the previous computation ensures $\chi(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 8$. Moreover, since $\overline{S}$ is of general type, $\operatorname{ext}^0(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 0$, and we obtain again

$$\operatorname{ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 8 \Leftrightarrow \operatorname{ext}^2(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 0\,.$$

2.3.7. REMARK. To show that $\Phi\colon \mathcal{M}_\nu \to \mathfrak{M}_\nu$ is an isomorphism, it is enough to prove that for every Godeaux surface S with torsion of order ν, we have:

(1) $\operatorname{ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 8$;
(2) Φ is bijective on tangent vectors.

Clearly $\mathcal{M}_\nu$ is smooth of dimension 8. The first condition ensures, by Remark 2.3.6, that also the moduli stack $\mathfrak{M}_\nu$ is so. Hence we can apply the criterion of Lemma 2.3.5: condition 2.3.5.1 is already proved in Lemma 2.3.4, while condition 2.3.5.2 is the second requirement listed here.

We will prove the two conditions in the case $\nu = 5$. In the following, we will write "Godeaux surface" for "Godeaux surface with torsion of order 5".

2.3.8. LEMMA. *Let $\overline{X} \subseteq \mathbb{P}^3$ be a quintic hypersurface with at most RDP as singularities; then $H^1(\overline{X}, T_{\mathbb{P}^n}|_{\overline{X}})$ vanishes.*

PROOF. We will prove that $H^1(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})^\vee = 0$. By adjunction, $\omega_{\overline{X}} = \mathcal{O}_{\overline{X}}(1)$; hence we can apply Serre duality to get the equality $H^1(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})^\vee = H^1(\overline{X}, \Omega_{\mathbb{P}^3}|_{\overline{X}}(1))$.

From the cohomology of the Euler sequence after tensoring by $\mathcal{O}_{\overline{X}}(1)$, we get

$$H^0(\overline{X}, \mathcal{O}_{\overline{X}}^{\oplus(n+1)}) \to H^0(\overline{X}, \mathcal{O}_{\overline{X}}(1)) \to \\ \to H^1(\overline{X}, \Omega_{\mathbb{P}^3}|_{\overline{X}}(1)) \to H^1(\overline{X}, \mathcal{O}_{\overline{X}}^{\oplus(n+1)})\,;$$

the first map is surjective, while, since $q(\overline{X}) = 0$, the last group is equal to $H^1(\overline{X}, \mathcal{O}_{\overline{X}})^{\oplus(n+1)} = 0$. Hence, $H^1(\overline{X}, \Omega_{\mathbb{P}^3}|_{\overline{X}}(1)) = 0$. □

2.3.9. LEMMA. *The moduli stack $\mathfrak{M}_5$ of Godeaux surfaces is smooth of dimension* 8.

PROOF. By Remark 2.3.6, it is enough to show $\mathrm{ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = 8$ for every S.

Let $X \to S$ be the cover associated to the torsion of S; we have seen that $\overline{X}$, the canonical model of X, embeds in $\mathbb{P}^3$ as a quintic hypersurface with at most RDP; also, $\mathrm{Ext}^i(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}})$ is just the $\mathbb{Z}_5$-invariant part of $\mathrm{Ext}^i(\Omega_{\overline{X}}, \mathcal{O}_{\overline{X}})$. Applying the functor $\mathrm{Hom}(,\mathcal{O}_{\overline{X}})$ to

$$0 \to \mathcal{O}_{\overline{X}}(-5) \to \Omega_{\mathbb{P}^3}|_{\overline{X}} \to \Omega_{\overline{X}} \to 0\,,$$

we get the exact sequence

$$\text{(6)}\quad \mathrm{Hom}(\Omega_{\overline{X}}, \mathcal{O}_{\overline{X}}) \to H^0(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}}) \to H^0(\overline{X}, \mathcal{O}_{\overline{X}}(5)) \to \\ \to \mathrm{Ext}^1(\Omega_{\overline{X}}, \mathcal{O}_{\overline{X}}) \to H^1(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})\,.$$

The first group is zero because $\overline{X}$ is of general type, while we already proved that the last one vanishes in Lemma 2.3.8. Therefore we have a short exact sequence and to compute $\mathrm{ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}})$ we observe that

(1) $H^0(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})$ has the same dimension as the group $\mathbb{P}\,GL(4)$, that is 15; its $\mathbb{Z}_5$-invariant part has dimension 3, since it parametrizes infinitesimal deformation of linear isomorphisms of $\mathbb{P}^3$ commuting with the action of G, and these correspond to diagonal matrices;

(2) $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(5))$ has dimension $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) - 1 = \binom{3+5}{5} - 1 = 55$; as we saw before, $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))^{\mathbb{Z}_5} = 12$, then $h^0(\overline{X}, \mathcal{O}_{\overline{X}}(5))^{\mathbb{Z}_5} = 11$.

In particular, we obtain that $\operatorname{ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}}) = \operatorname{ext}^1(\Omega_{\overline{X}}, \mathcal{O}_{\overline{X}})^{\mathbb{Z}_5} = 11 - 3 = 8$. $\square$

2.3.10. LEMMA. *The morphism Φ is bijective on tangent vectors.*

PROOF. Fix a Godeaux surface S. Then $T_{\mathfrak{M}_5,[\overline{S}]} = \operatorname{Ext}^1(\Omega_{\overline{S}}, \mathcal{O}_{\overline{S}})$, while $T_{\mathcal{M}_5,[S]} = T_{\tilde{M}_5,[S]}$, since the projection $\tilde{M}_5 \to \mathcal{M}_5$ is an étale cover. The morphism between the tangent spaces induced by Φ is the restriction, first to the $\mathbb{Z}_5$-invariant part, then to $T_{\tilde{M}_5,[S]}$, of the map

$$H^0(\overline{X}, \mathcal{O}_{\overline{X}}(5)) \to \operatorname{Ext}^1(\Omega_{\overline{X}}, \mathcal{O}_{\overline{X}})$$

in the exact sequence (6).

If f is the quintic polynomial defining $\overline{X}$, $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(5))^{\mathbb{Z}_5}$, as the restriction of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))^{\mathbb{Z}_5}$, consists of quintic polynomials invariant with respect to the action of G; but they can be interpreted also as infinitesimal deformations of f inside the quintic polynomials invariant with respect to G. In the same spirit, $H^0(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})^{\mathbb{Z}_5}$ is the space of infinitesimal deformations of the identity matrix inside the matrices invariants with respect to G, that are the diagonal matrices.

In other words, an element of $H^0(\overline{X}, T_{\mathbb{P}^3}|_{\overline{X}})^{\mathbb{Z}_5}$ can be represented by an infinitesimal deformation $I + \varepsilon A$ with A a diagonal matrix, modulo multiplication with scalars; to this, we associate an infinitesimal deformation of polynomials $(I + \varepsilon A)f$, represented by the polynomial Af in the space $H^0(X, \mathcal{O}_{\mathbb{P}^3}(5))^{\mathbb{Z}_5}$. Since A is diagonal, Af has exactly the same monomials of f, only with different coefficients, and in particular it is of the form $\sum a_i x_i^5 + \cdots$ with $a_i \neq 0$, and therefore does not intersect $T_{\tilde{M}_5,[S]}$, that contains only monomials without the terms x_i^5. $\square$

The last two lemmas, in view of Remark 2.3.7, yield the following theorem.

2.3.11. THEOREM. *The morphism $\Phi\colon \mathcal{M}_5 \to \mathfrak{M}_5$ is an isomorphism of stacks.*

2.2.4 *Inertia stacks*

In this section we will compute the inertia stack of $\mathcal{M}_\nu$ for $\nu \in \{3, 4, 5\}$. Since $\mathcal{M}_3$ has trivial automorphism groups (i.e., it is an algebraic space), we will work only on $\mathcal{M}_4$ and $\mathcal{M}_5$. These are quotients of an open subscheme of $\mathbb{A}^8$ by an explicit finite group of projective matrices. Hence we can work out the components of the inertia stacks $I(\mathcal{M}_4)$ and $I(\mathcal{M}_5)$ from these representations.

2.2.4.1 Torsion of order five. Let us have a look at Figure 1; our problem is to identify automorphisms of Godeaux surfaces lying in different subspaces of $\tilde{M}_5$. In the following, we will denote a generic surface in $\tilde{M}_5$ as $S_{\tilde{M}_5}$; in the same way, we define S_Q, S_P, S_H, S_O.

In this case we do not need many computations: for example, there is a unique way to identify $\mathrm{Aut}(S_Q) \cong \mathbb{Z}_2$ inside $\mathrm{Aut}(S_P) \cong \mathbb{Z}_4$; the only ambiguities come up when we want to see where $\mathrm{Aut}(S_P) \cong \mathbb{Z}_4$ and $\mathrm{Aut}(S_H) \cong \mathbb{Z}_5$ goes inside $\mathrm{Aut}(S_O) \cong \mathbb{Z}_5^2 \ltimes \mathbb{Z}_4$. These are not actual problems, since to construct $\mathrm{I}(\mathcal{M}_5)$ we only need to see which automorphisms go to coincide when viewed in a larger group. Then it is obvious that only the identities will coincide in $\mathrm{Aut}(S_0)$, since the other automorphisms have different orders.

In order to explain the general principle, we will give the computation even if it is not really necessary. In the following, we will write all groups $\mathrm{Aut}(S)$ as subgroup of the group $\mathrm{Aut}(S_0)$; this one is the quotient by $G \cong \mathbb{Z}_5$ of the group $H_5 \cong \mathbb{Z}_5^3 \ltimes \mathbb{Z}_5^\star$. We will denote the matrix

$$\mathrm{diag}(1, \zeta^{i_2}, \zeta^{i_3}, \zeta^{i_4}) P_{\sigma^h} \in H_\nu \,,$$

where σ is the permutation $(2,1,3,4)$, with (i_2, i_3, i_4, h); G lies inside H_ν as the subgroup generated by $(1,2,3,0)$. The same program we used to compute the abstract automorphism groups gives us also the automorphism groups embedded in $\mathbb{P}\,\mathrm{GL}(4)$; in particular we obtain the following representations in H_ν/G:

$$\begin{aligned}
\mathrm{Aut}(S_{\mathcal{M}_5}) &= \langle (0,0,0,0) \rangle \,, \\
\mathrm{Aut}(S_Q) &= \langle (0,0,0,2) \rangle \,, \\
\mathrm{Aut}(S_P) &= \langle (0,0,0,1) \rangle \,, \\
\mathrm{Aut}(S_H) &= \langle (0,0,1,0) \rangle \,, \\
\mathrm{Aut}(S_O) &= \langle (1,0,0,0), (0,1,0,0), (0,0,0,1) \rangle \,.
\end{aligned}$$

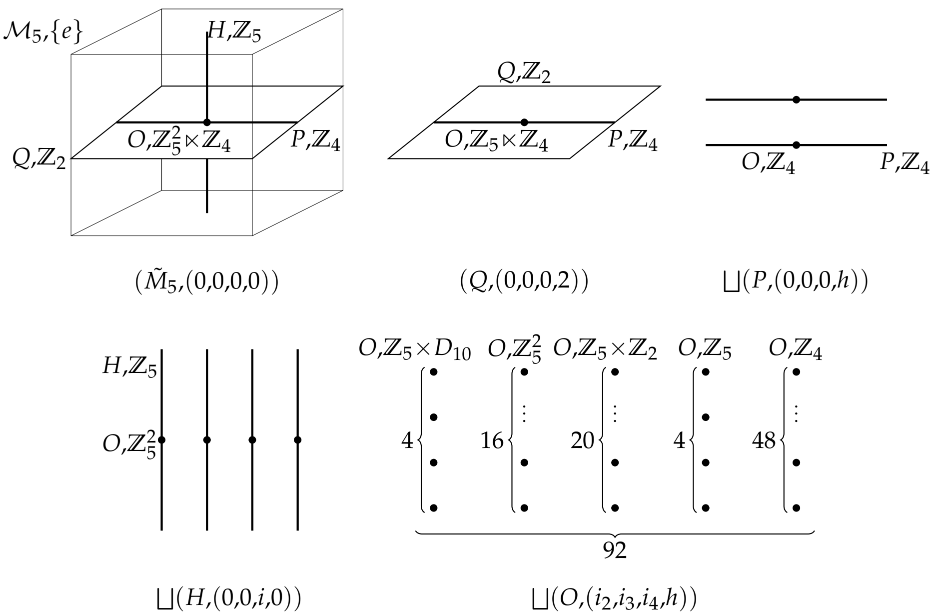
Note that these are the embedded automorphism groups for just one component of H, P and Q: indeed, if we do not choose carefully the components we may end with incompatible groups. We just have to do a choice of components that satisfies the Hasse diagram of containments even in $\tilde{M}_5$.

Once we have this explicit description, we know how automorphisms glue amongst different subschemes of $\tilde{M}_5$, and we can write down the 100 components of $\mathrm{I}(\mathcal{M}_5)$:

$$\mathrm{I}(\mathcal{M}_5) = (\mathcal{M}_5, (0,0,0,0)) \sqcup (Q, (0,0,0,2)) \sqcup \bigsqcup_{h\in\{1,3\}} (P, (0,0,0,h)) \sqcup$$
$$\sqcup \bigsqcup_{i\in\{1,2,3,4\}} (H, (0,0,i,0)) \sqcup \bigsqcup_{i_1,i_2,i_3,h} (O, (i_1, i_2, i_3, h)) \,,$$

where the last union runs over all the 92 elements of $\mathrm{Aut}(S_0)$ not previously considered. We can find automorphism groups of all subcomponents of the components of the inertia stack by computing centralizers. For example, the automorphism group of $O \subseteq (Q, (0,0,0,2))$ is the centralizer of $(0,0,0,2)$ inside H_v/G, that is $\mathbb{Z}_5 \ltimes \mathbb{Z}_5^\star$. It is easy to use *GAP* to compute all centralizers of each automorphism inside H_v/G (we do not need the other centralizers since all other groups are abelian and so the centralizers are trivial).

The following picture represents all the components of the inertia stack with all their stacky subcomponents (obviously with fake dimensions).



In particular we observe that the special point O inside the component $(P, (0,0,0,h))$ is not really special, since its automorphism group is the same as the one of P.

2.2.4.2 Torsion of order four. We proceed in the same way as before, using Figure 2. This time, all automorphisms live in the subgroup $H_4 \cong \mathbb{Z}_4^3 \ltimes \mathbb{Z}_2$ of $\mathbb{P}\,\mathrm{GL}(8)$. The isomorphism associates to (i_1, i_3, j_1, h) the matrix

$$\mathrm{diag}(\xi^{2i_1}, 1, \xi^{2i_3}, \xi^{i_3}, \xi^{i_1+i_3}, \xi^{i_1}, \xi^{j_1}, \xi^{-j_1}) P_{\sigma^h} \,,$$

where σ is the permutation $(1,3)(4,6)(7,8)$. Inside H_4, G is generated by $(1,3,1,0)$.

We obtain the following presentation in H/G:

$$\begin{aligned}
\mathrm{Aut}(S_{\mathcal{M}_4}) &= \langle (0,0,0,0) \rangle \,, \\
\mathrm{Aut}(S_{R_1}) &= \langle (2,2,0,0) \rangle \,, \\
\mathrm{Aut}(S_W) &= \langle (0,0,0,1) \rangle \,, \\
\mathrm{Aut}(S_{R_4}) &= \langle (2,2,0,0), (0,0,0,1) \rangle = \langle \mathrm{Aut}(S_W), \mathrm{Aut}(S_{R_1}) \rangle \,, \\
\mathrm{Aut}(S_{R_{2,3}}) &= \langle (0,2,0,0) \rangle \,, \\
\mathrm{Aut}(S_{S_4}) &= \langle (1,1,0,0) \rangle \,, \\
\mathrm{Aut}(S_{S_5}) &= \langle (2,2,0,0), (0,2,0,0) \rangle = \langle \mathrm{Aut}(S_{R_1}), \mathrm{Aut}(S_{R_{2,3}}) \rangle \,, \\
\mathrm{Aut}(S_{S_6}) &= \langle (1,1,0,0), (0,0,0,1) \rangle = \langle \mathrm{Aut}(S_W), \mathrm{Aut}(S_{S_4}) \rangle \,, \\
\mathrm{Aut}(S_{S_7}) &= \langle (2,2,0,0), (0,2,0,0), (0,0,0,1) \rangle = \\
&= \langle \mathrm{Aut}(S_W), \mathrm{Aut}(S_{S_5}) \rangle \,, \\
\mathrm{Aut}(S_{S_{2,3}}) &= \langle (0,1,0,0) \rangle \,, \\
\mathrm{Aut}(S_{T_{2,3}}) &= \langle (0,1,0,0), (2,2,0,0) \rangle = \langle \mathrm{Aut}(S_{S_4}), \mathrm{Aut}(S_{R_1}) \rangle \,.
\end{aligned}$$

Now we can write the inertia stack:

$$\begin{aligned}
\mathrm{I}(\mathcal{M}_4) = (\mathcal{M}_4, (0,0,0,0)) \sqcup (R_1, (2,2,0,0)) \sqcup (W, (0,0,0,1)) \sqcup \\
\sqcup (R_{2,3}, (0,2,0,0)) \sqcup (R_4, (2,2,0,1)) \sqcup (S_5, (0,2,2,0)) \sqcup \\
\sqcup \bigsqcup_{i\in\{1,3\}} (S_4, (i,i,0,0)) \sqcup \bigsqcup_{i\in\{1,3\}} (S_{2,3}, (0,i,0,0)) \sqcup \bigsqcup_{i\in\{1,3\}} (S_6, (i,i,0,1)) \sqcup \\
\sqcup \bigsqcup_{i\in\{0,2\}} (S_7, (0,2,i,1)) \sqcup \bigsqcup_{i\in\{1,3\}} (T_{2,3}, (2,i,0,0)) \,.
\end{aligned}$$

We do not try to draw the components, since there are many more than in the case $\nu = 5$ and more scattered through the various dimensions. We still have to show what are the automorphism groups of the subcomponents of the components of the inertia stack. Again, these are trivially the original automorphism groups if this is abelian; so the only case to study is S_7. Table 4 sums up the results gathered with a *GAP* program similar to the one previously used.

2.3 General bounds on the automorphism groups

In the previous section we computed the automorphism groups of all Godeaux surfaces with torsion of order at least 3. We know that the other two classes of Godeaux surfaces, with torsion isomorphic

Component	$\mathrm{Aut}(S_7)$
$(\mathcal{M}_4, (0,0,0,0))$	D_8
$(R_1, (2,2,0,0))$	D_8
$(W, (0,0,0,1))$	$\mathbb{Z}_2^2$
$(R_{2,3}, (0,2,0,0))$	$\mathbb{Z}_2^2$
$(R_4, (2,2,0,1))$	$\mathbb{Z}_2^2$
$(S_5, (0,2,2,0))$	$\mathbb{Z}_2^2$
$(S_7, (0,2,2,1))$	$\mathbb{Z}_4$
$(S_7, (0,2,0,1))$	$\mathbb{Z}_4$

TABLE 4. automorphism groups for the S_7 subcomponents.

to $\mathbb{Z}_2$ and with no torsion, are non-empty, but there is not a complete classification of these surfaces.

In order to work towards a classification, we can direct our attention to the study of specific properties that such surfaces are required to have. In this section, we compute an estimate of the number of automorphisms of Godeaux surfaces using their intrinsic properties. Moreover, we study the structure that the automorphism groups must have. We will study the case of Godeaux surfaces with bicanonical system with no fixed part. It is worth mentioning that in the literature there are no known example of the other case, that is, of Godeaux surfaces with a non-trivial fixed component in the bicanonical system.

As previously said, Xiao's estimate for a generic surface of general type is $|\mathrm{Aut}(S)| \leq 42^2 K_S^2$, and this gives, for a Godeaux surface S, $|\mathrm{Aut}(S)| \leq 1764$. This compares badly to the maximum number of automorphisms actually found in the previous section, that is 100 automorphisms for the classical Godeaux surface, given by the quotient of the Fermat quintic in $\mathbb{P}^3$.

Our results are collected in table 8 and for some cases are significantly smaller, despite not reaching the effective data computed in the previous sections for surfaces with big torsion.

2.3.1 *Preliminaries*

2.3.1.1 Machinery. Our main tool in this study will be the bicanonical fibration of a Godeaux surface. In this section we present some properties of fibrations (in particular onto $\mathbb{P}^1$) that will be useful in the future.

3.1.1. LEMMA. *Let $\varphi\colon S \to \mathbb{P}^1$ be a fibration (with reduced fibers), s the number of singular fibers, and H be the image of a finite subgroup of* $\mathrm{Aut}(S)$ *into* $\mathrm{Aut}(\mathbb{P}^1)$*. If H is isomorphic to the cyclic group C_r or to the dihedral group (with $2r$ elements) D_r, then $r \leq s$.*

PROOF. By **[Bea81]**, there are at least 3 singular fibers. Also, it is well known that the cyclic group C_r acting on $\mathbb{P}^1$ has just two non-trivial orbits, each with one point, and the dihedral group D_r acts on $\mathbb{P}^1$ with 3 non-trivial orbits: one with 2 points and the other two with r points.

Using these facts, we observe that there is at least one singular fiber over a point whose orbit has at least r elements. Hence, we have at least r singular fibers (over these points), and in particular $r \leq s$. □

3.1.2. REMARK. By the universal property of the blow up, there is an identification between automorphisms of a surface fixing a smooth point and automorphisms of the blow up surface fixing the exceptional curve. This identification can be easily extended to the case of multiple blow ups of different points, and also when we blow up infinitely near points. In this case, the automorphisms on the original surface must fix a point and a tangent vector, while the automorphisms on the blow up must fix the two exceptional curves without exchanging them.

3.1.3. LEMMA. *Let $f\colon S \to C$ be a fibration over a smooth curve, and s be the number of singular fibers of f. Then $s \leq \chi_{\mathrm{top}}(S) - 4(g-1)(b-1)$, where g and b are respectively the geometric genus of a smooth fiber and of the base.*

PROOF. Lemma VI.4 of **[Bea96]**, we have

$$\chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(C)\chi_{\mathrm{top}}(F) + \sum_{F'}(\chi_{\mathrm{top}}(F') - \chi_{\mathrm{top}}(F)), \tag{7}$$

where F is a smooth fiber and the sum runs over all fibers (equivalently, all singular fibers). But then

$$s \leq \sum(\chi_{\mathrm{top}}(F') - \chi_{\mathrm{top}}(F)) = \chi_{\mathrm{top}}(S) - 4(g-1)(b-1). \qquad \square$$

2.3.1.2 **Numerical Godeaux surfaces.** Let S be a (numerical) Godeaux surface. The generic curve $C \in \Lambda := |2K_S|$ is a connected (every n-canonical divisor is 1-connected) genus 4 curve, because the genus formula says

$$g(2K_S) = 1 + \frac{1}{2}((2K_S)^2 + (2K_S)\cdot K_S) = 4.$$

If there is no fixed part in the bicanonical linear system, then $|2K_S|$ has at most $(2K_S)^2 = 4$ base points. More precisely, there are these possibilities: four base points of multiplicities $(1,1,1,1)$; three base points of multiplicities $(2,1,1)$; two base points of multiplicities $(2,2)$; two base points of multiplicities $(3,1)$; one base points of multiplicity (4). By Bertini's theorem, the general member of $|2K_S|$ is smooth away from the base points. By numerical reasons, it is smooth also on the base points in all cases except the last one, where it can be smooth or have a double point. In this last case, the genus of the normalization is hence 3.

If there is a fixed part in the bicanonical pencil, write $|2K_S| = |M| + F$; then by **[CP00]** we have only one possibility: $F^2 = -2$, $M^2 = 2$, $M \cdot F = 2$, with $g(M) = 3$.

Putting all together, these are the possibilities:

(1) the bicanonical system is without fixed divisor and:
   (a) its generic member is smooth with:
      (i) four base points of multiplicities $(1,1,1,1)$;
      (ii) three base points of multiplicities $(2,1,1)$;
      (iii) two base points of multiplicities $(2,2)$;
      (iv) two base points of multiplicities $(3,1)$;
      (v) one base points of multiplicity (4);
   (b) there is one base point where the generic member has a double point;
(2) the bicanonical system can be written as $|2K_S| = F + |M|$ with $F^2 = -2$, $M^2 = 2$, $M \cdot F = 2$, and $g(M) = 3$.

We recall that the topological Euler characteristic for a Godeaux surface S is $\chi_{\text{top}}(S) = 11$, and that blowing up a smooth point of a surface increases its topological Euler characteristic by 1.

2.3.2 *Assuming no fixed part*

2.3.2.1 General philosophy. Given S a Godeaux surface with no fixed part in the bicanonical system, we can blow up the base points enough times to obtain a surface $\hat{S}$ with a bicanonical system which is a fibration over $\mathbb{P}^1$ with smooth fibers, as in table 5.

Once we blow up enough times, we get a fibration $\hat{\varphi}\colon \hat{S} \to \mathbb{P}^1$, and we can apply Lemma 3.1.3 to have an estimate on the number of singular fibers of $\hat{\varphi}$. Since $(2K_S)^2$ is still 4, the genus of the generic fiber F is 4 by the genus formula, hence we get that the number of singular fibers is at most $\chi_{\text{top}}(\hat{S}) - 4(g-1)(b-1) = \chi_{\text{top}}(\hat{S}) + 12 = \chi_{\text{top}}(S) + \#\text{blow ups} + 12 = 23 + \#\text{blow ups}$.

Base points	Type	General fiber	Blow ups
4	$(1,1,1,1)$	smooth	4
3	$(2,1,1)$	smooth	4
2	$(2,2)$	smooth	4
2	$(3,1)$	smooth	4
1	(4)	smooth	4
1	(4)	singular	1

TABLE 5. Blow ups needed to obtain a fibration. In all cases we need 4 blow ups, but in the last one we need just 1.

Since the fibration is canonical, every automorphism of $\hat{S}$ is compatible with $\hat{\varphi}$. In particular, we have an exact sequence of groups

$$1 \longrightarrow \underset{\substack{\cap\!|\\ \mathrm{Aut}(F)}}{K} \longrightarrow \underset{\substack{\wr|\!|\\ \mathrm{Aut}(\hat{S})}}{G} \longrightarrow \underset{\substack{\cap\!|\\ \mathrm{Aut}(\mathbb{P}^1)}}{H} \longrightarrow 1\,, \tag{8}$$

where H is the image of G in $\mathrm{Aut}(\mathbb{P}^1)$ and K is the kernel of this homomorphism. In other words, K contains the automorphisms of $\hat{S}$ that fix every fiber of φ. Hence, any element of K can be viewed as an element of $\mathrm{Aut}(F)$ for every fiber F of $\hat{\varphi}$. The group K can be identified with a subgroup of $\mathrm{Aut}(F)$ as long as we consider a generic F (if this was not true, the starting automorphism was forced to be the identity in $\mathrm{Aut}(\hat{S})$).

Now, by Iitaka's theorem, $\mathrm{Aut}(S) = \mathrm{Aut}(\hat{S})$ is finite, hence H is finite. There are two family and three sporadic possibilities for H: cyclic and dihedral groups of any order, the symmetric group on four elements, or the alternating groups on four or five elements. Also, by Lemma 3.1.1, if $H \cong C_r$ or $H \cong D_r$, then $|H|$ is less or equal than the number s of singular fibers of $\hat{\varphi}$. By **[Pal08]**, $3 \nmid |G|$, hence $3 \nmid |H|$; this rules out the sporadic groups and the cyclic or dihedral groups of order divisible by 3.

3.2.1. REMARK. By **[CCML07]**, an involution of a Godeaux surface is composed with the bicanonical system, that is, an involution does not permutes the fibers of the bicanonical fibration, hence it descends on $\mathbb{P}^1$ as the trivial automorphism.

This does not imply that the subgroup H of automorphisms of $\mathbb{P}^1$ has no involutions (equivalently, has odd order), because an involution in H may exist as long as it lifts to an element of order different from 2 in G. But we can deduce that if $|K|$ is odd, then also $|H|$ is forced to be odd, and therefore $|G|$ is odd or it is divisible by 4.

2.3.2.2 No fixed part and general fiber smooth.

Let us restrict to the case when the general member of $|2K_S|$ is smooth (we treat separately the case in which there is a single base point of multiplicity 4, where the general member has a singularity of multiplicity 2). In this case we need 4 blow ups to obtain $\hat{S}$, hence the number of singular fibers is at most 27; so, H can be C_r or D_r with $1 \leq r \leq 27$ and $3 \nmid r$, and the maximum order for H is $26 \cdot 2 = 52$.

On the other side, for the generic fiber F we have $K \leq \mathrm{Aut}(F)$; a generic fiber is smooth, and by Hurwitz we have $|\mathrm{Aut}(F)| \leq 42 \deg K_F = 252$. But it is known that there are no Hurwitz curves of genus 4. More precisely, automorphism groups of genus 4 curves are classified in **[MSSV02]**, and their cardinalities are among

$$\{120, 72, 40, \ldots\}.$$

Now, by **[Pal08]**, $3 \nmid |G|$, hence $3 \nmid |K|$, so K has order at most 40. But the curves with 40 automorphisms are rigid (more precisely, they have no deformations with the same number of automorphisms). This implies that if that was the case, the bicanonical fibration would induce an isotrivial fibration on S.

If we assume that this is not true, then $|K|$ is bounded by the cardinality of the biggest group of automorphisms of a smooth genus 4 curve whose locus in M_4 has dimension at least 1. This cardinality is 20, by **[MSSV02]**.

Hence, we end up with the estimate

$$|\mathrm{Aut}(S)| = |K| \cdot |H| \leq \begin{cases} 20 \cdot 52 = 1040 & \text{if } |2K_S| \text{ does not induce} \\ & \text{an isotrivial fibration,} \\ 40 \cdot 52 = 2080 & \text{if } |2K_S| \text{ induces an} \\ & \text{isotrivial fibration.} \end{cases}$$

A better estimate can be found with another observation. Take a base point of the bicanonical system, say, p, and consider the subgroup, maybe not normal, $G_p \leq G$ of automorphisms of S that fix p. We have $[G : G_p] = |G \cdot p| \in \{1, 2, 4\}$, depending on the shape of the base locus of $|2K_S|$. If we estimate G_p, we automatically get an estimate for G too.

If G_p fixes p, it acts on its tangent space, and we can use the same procedure as before: we build an exact sequence

$$1 \to K_p \to G_p \to H_p \to 1\,,$$

where H_p is the image of G_p on $\mathrm{Aut}(\mathbb{P}^1)$: we can think of this $\mathbb{P}^1$ in three ways: as the base of the fibration; as the projectivization of the tangent space of the base point, blown up enough times to make it of multiplicity one; finally, as the exceptional divisor of the base point blown up one time more.

There are two main ways in which using G_p instead of G helps: the first is that, by Cartan's Lemma, and in particularly by Corollary 3.2.3, we can restrict to the case in which K_p is cyclic; the second is that when p has multiplicity at least 2, then H_p acts on $\mathbb{P}^1$ either in the same way in which $H_p \leq H$ acts on the base of the fibration, but also with a fixed point (the tangent direction determined by any 2-canonical curve passing through p). Again by Cartan's Lemma, a group acting on $\mathbb{P}^1$ stabilizing a point must be a subgroup of $\mathbb{C}^\star$, hence cyclic if finite.

3.2.2. Lemma (Cartan). *Let (S, p) a scheme with a marked point, with tangent space $T = T_pS$, and let G_p a finite group of automorphisms of (S, p). Then there is an associate representation $G_p \hookrightarrow \mathrm{GL}(T)$ which is faithful.*

3.2.3. Corollary. *If S is a surface of general type, consider the composition morphism $G_p \hookrightarrow \mathrm{GL}(T) \to \mathrm{Aut}(\mathbb{P}^1)$. Its kernel is a finite subgroup of $\mathbb{C}^\star$, hence it is cyclic.*

So we consider the image Γ of G in the permutation group Σ_l ($l \leq 4$ is the number of base points): by **[Pal08]** there are no order three elements in the image, so the subgroup of Σ_l can be $\{1\}$, C_2 (two ways), C_2^2 (two ways), C_4, D_8. For each subgroup, up to conjugacy and point renaming, we try fixing a certain number of points (at least one), obtaining a subgroup $\Gamma' \leq \Gamma$, the stabilizer of the selected points.

Let $G' \leq G$ be the subgroup of automorphisms of S behaving on the base points as an element of Γ'. We look at the sequence

$$1 \to K_p \to G' \to H_p \to 1\,,$$

where p is one of the points fixed by G'. As said before, K_p is cyclic, while $|H_p|$ is bounded by 52 in the even case (D_{26}) and by 25 in the odd case (C_{25}). An additional condition is given by Remark 3.2.4, that reduces $|H_p|$ to at most C_{26} when p has multiplicity at least 2.

3.2.4. Remark. Suppose that H_p is the image on $\mathrm{Aut}(\mathbb{P}^1)$ of a group acting on S fixing the base point p and that p is of multiplicity at least 2. Then all the curves in $|2K_S|$ pass through p with the same tangent direction. The group H_p, must fix this direction with all its elements, hence cannot be dihedral.

The stabilizer Γ' gives the residual action of Γ on the other points, and prescribe the minimum ramification that the action must induce on the quotient F/K_p. On the other hand, $[\Gamma : \Gamma'] = [G : G']$, hence the estimate is given by $|G| = |G'| \cdot [G : G'] = |H_p| \cdot |K_p| \cdot [\Gamma : \Gamma']$.

In the case we have 4 simple base points, we can prove the following.

3.2.5. Proposition. *Let S be a Godeaux surface with* 4 *simple base points, and let p be a prime such that C_p acts faithfully on S. Then $p \in \{2, 5\}$.*

Proof. As always, we need to prove that $p \leq 5$, since the case $p = 3$ is already ruled out. Suppose that $C_p = \langle g \rangle$ with $p > 5$ acts on S; then all base points are fixed by g. Moreover, the generic fiber is smooth, and if f fixed it, we would have a faithful action of C_p on a smooth curve of genus 4 with 4 fixed points, and this is impossible by Riemann-Hurwitz (the biggest possible p is 5).

Hence, C_p acts permuting the fibers of the fibration, and sends two fibers in themselves: F_0 and F_∞. Consider a base point $x \in S$ and consider $A_0 \subseteq F_0$ and $A_\infty \subseteq F_\infty$, the irreducible components of the two fibers passing through x. If g acted trivially on both, g would be the identity in the tangent space at x, being the identity in two orthogonal directions. Hence there exists A, an irreducible component of a fiber, passing through at least a base point and on which g is not trivial.

We have two possibilities: either A passes through 4 base points (that is, $K_S \cdot A = 2$), or through 2 base points (and $K_S \cdot A = 1$).

In the first case, A has arithmetic genus less or equal than 4, and g fixes 4 points; this is not possible for the same reason we saw before.

In the second case, $K_S \cdot A = 1$, and $p_a(A) = (A^2 + 3)/2$; we also have

$$2 = 2K_S \cdot A = F_i \cdot A = (A + F_i \setminus A) \cdot A\,,$$

hence $A^2 \leq 2 - A \cdot (F_i \setminus A)$; but $2K_S$ is 1-connected, hence $A^2 \leq 1$, and $p_a(A) \leq 2$. On the other hand, $p_a(A) \geq 0$ implies $A^2 \geq -3$ and $1 \leq A \cdot (F_i \setminus A) \leq 2 - A^2 \leq 5$, therefore these points must be also fixed by g. Summing up, we have an action of C_p on a curve of genus at most 2, with at least 3 fixed points, and this is again not possible by Riemann-Hurwitz (the highest possible p is 5). □

The results are collected in table 6. The table is split into cases depending on the number of base points, their multiplicities, and the action of G on the base points. Then we have the additional choice of Γ', that is, on the base points we choose to fix to compute the estimate; but, for every case, we can consider only the Γ' that give us the best estimate. Recall that, in order to obtain better estimates, we used Remark 3.2.1 to split the result in two parts, depending on the parity of r (that is, of the order of the induced group of automorphisms of $\mathbb{P}^1$).

The orders r that are underlined in the table are the ones that can act only on a finite number of non-isomorphic genus 4 curves, as stated in **[MSSV02]**.

See also section 3.3 for the details on the cyclic actions on a curve.

2.3.2.3 No fixed part and singular general fiber.

The only case that fits in this description is when there is a single base point of $|2K_S|$ on which the general bicanonical curve has a simple node. There are exactly two special bicanonical curves.

Estimate of $|H|$. A single blow up is sufficient to obtain a fibration whose general fiber is a smooth curve of genus 3. The number of singular fiber is then at most $\chi_{\text{top}}(S) + 1 - 4(g-1)(b-1) = 11 + 1 - 4 \cdot 2 \cdot (-1) = 20$ and so the maximum order of H is 40, realized for D_{20}.

Estimate of $|K|$. Let p be the base point of S; of course $G = G_p$, and as before we can use Cartan's Lemma to see G inside $\mathrm{GL}(T_pS)$ and eventually in $\mathrm{Aut}(E)$, where E is also the exceptional curve in $\hat{S}$. There is a 2 to 1 morphism $E \to \mathbb{P}^1$, where the target is the base of the bicanonical fibration. Let $\sigma\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the involution that realizes the 2 to 1 morphism: by Riemann-Hurwitz, it has 2 fixed points, that again must correspond to two double curves in the system $|2K_s|$.

Let H' be the image of G inside $\mathrm{Aut}(E)$ and consider the exact sequence

$$1 \to K' \to G \to H' \to 1\,.$$

The kernel K' contains all automorphisms of S that send all fibers of $|2K_S|$ to themselves, but also preserve the two branches of each fiber passing through the base point. We have that $[K : K'] \leq 2$ (where K is as always the subgroup of G with the only condition of fixing the fibers). By Cartan's lemma, K' is cyclic, and it acts faithfully on the normalization of the generic curve (of genus 3) with two fixed points,

B.p.	Γ	$\lvert\Gamma\rvert$	Γ'	$[\Gamma : \Gamma']$	Ram.	Tab.	r	r even	r odd	Max	Min
$(1,1,1,1)$	$1 = \{1\}$	1	$\Gamma_1 = \{1\}$	1	$(\bullet)^4$	9	$1^{a,f}, 2, 4, 5^b$	$1 \cdot 4 \cdot 50 = 200$	$1 \cdot 5 \cdot 25 = 125$	200	200
	$C_2 = \langle (3,4) \rangle$	2	$\Gamma_1 = \langle (3,4) \rangle$	1	$(\bullet)^2(\bullet\bullet)$	10	$2, 4, 10$	$1 \cdot 10 \cdot 50 = 500$	—	500	400
			$\Gamma_3 = \{1\}$	2	$(\bullet)^4$	9	$1, 2, 4, 5$	$2 \cdot 4 \cdot 50 = 400$	$2 \cdot 5 \cdot 25 = 250$	400	
	$C_2 = \langle (1,2)(3,4) \rangle$	2	$\Gamma_1 = \{1\}$	2	$(\bullet)^4$	9	$1^c, 2, 4, 5$	$2 \cdot 4 \cdot 50 = 400$	$2 \cdot 5 \cdot 25 = 250$	400	400
	$C_2^2 = \langle (1,2), (3,4) \rangle$	4	$\Gamma_1 = \langle (3,4) \rangle$	2	$(\bullet)^2(\bullet\bullet)$	10	$2, 4, 10$	$2 \cdot 10 \cdot 50 = 1000$	—	1000	800
			$\Gamma_{1,3} = \{1\}$	4	$(\bullet)^4$	9	$1, 2, 4, 5$	$4 \cdot 4 \cdot 50 = 800$	$4 \cdot 5 \cdot 25 = 500$	800	
	$C_2^2 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$	4	$\Gamma_1 = \{1\}$	4	$(\bullet)^4$	9	$1, 2, 4, 5$	$4 \cdot 4 \cdot 50 = 800$	$4 \cdot 5 \cdot 25 = 500$	800	800
	$C_4 = \langle (1,2,3,4) \rangle$	4	$\Gamma_1 = \{1\}$	4	$(\bullet)^4$	9	$1^d, 2, 4, 5^e$	$4 \cdot 4 \cdot 50 = 800$	$4 \cdot 5 \cdot 25 = 500$	800	800
	$D_8 = \langle (1,2,3,4), (1,3) \rangle$	8	$\Gamma_1 = \langle (2,4) \rangle$	4	$(\bullet)^2(\bullet\bullet)$	10	$2, 4, 10$	$4 \cdot 10 \cdot 50 = 2000$	—	2000	1600
			$\Gamma_{1,2} = \{1\}$	8	$(\bullet)^4$	9	$1, 2, 4, 5$	$8 \cdot 4 \cdot 50 = 1600$	$8 \cdot 5 \cdot 25 = 1000$	1600	
$(2,1,1)$	$1 = \{1\}$	1	$\Gamma_1 = \{1\}$	1	$(\bullet)^3$	11	$1, 2, 4, 5$	$1 \cdot 4 \cdot 26 = 104$	$1 \cdot 5 \cdot 25 = 125$	125	125
	$C_2 = \langle (2,3) \rangle$	2	$\Gamma_1 = \langle (2,3) \rangle$	1	$(\bullet)(\bullet\bullet)$	12	$2, 4, 10$	$1 \cdot 10 \cdot 26 = 260$	—	260	260
			$\Gamma_2 = \{1\}$	2	$(\bullet)^3$	11	$1, 2, 4, 5$	$2 \cdot 4 \cdot 52 = 416$	$2 \cdot 5 \cdot 25 = 250$	416	
$(3,1)$	$1 = \{1\}$	1	$\Gamma_1 = \{1\}$	1	$(\bullet)^2$	13	$1, 2, 4, 5, 8, 10, \underline{16}$	$1 \cdot 16 \cdot 26 = 416$	$1 \cdot 5 \cdot 25 = 125$	416	416
$(2,2)$	$1 = \{1\}$	1	$\Gamma_1 = \{1\}$	1	$(\bullet)^2$	13	$1, 2, 4, 5, 8, 10, \underline{16}$	$1 \cdot 16 \cdot 26 = 416$	$1 \cdot 5 \cdot 25 = 125$	416	416
	$C_2 = \langle (1,2) \rangle$	2	$\Gamma_1 = \{1\}$	2	$(\bullet)^2$	13	$1, 2^g, 4, 5, 8, 10, \underline{16}$	$2 \cdot 16 \cdot 26 = 832$	$2 \cdot 5 \cdot 25 = 250$	832	832
(4)	$1 = \{1\}$	1	$\Gamma_1 = \{1\}$	1	$(\bullet)$	14	$1, 2, 4, 5, 8, 10, \underline{16}$	$1 \cdot 16 \cdot 26 = 416$	$1 \cdot 5 \cdot 25 = 125$	416	416

Table 6. Results in the case that the general member of $\lvert 2K_S \rvert$ is smooth.

Ref.	Tors(S)	Aut(S)	$\lvert\mathrm{Aut}(S)\rvert$	Estimate	Dim.	Example equation
a	5	1	$1 \cdot 1 \cdot 1 = 1$	208	8	Generic
b	5	C_5	$1 \cdot 5 \cdot 1 = 5$	208	2	$\sum_{1 \le i \le 4} x_1^5 + b_1 x_2 x_3^3 x_4 + c_1 x_2^2 x_3 x_4^2 = 0$
c	5	C_2	$2 \cdot 1 \cdot 1 = 2$	416	4	$\sum_{1 \le i \le 4} x_1^5 + b(x_2 x_3^3 x_4 + x_1 x_2^3 x_3) + b'(x_1^3 x_3 x_4 + x_1 x_2 x_4^3) + c(x_2^2 x_3 x_4^2 + x_1^2 x_2 x_3^2) + c'(x_1 x_3^2 x_4^2 + x_1^2 x_2^2 x_4) = 0$
d	5	C_4	$4 \cdot 1 \cdot 1 = 4$	832	2	$\sum_{1 \le i \le 4} x_1^5 + b(x_2 x_3^3 x_4 + x_1^3 x_3 x_4 + x_1 x_2 x_4^3 + x_1 x_2^3 x_3) + c(x_2^2 x_3 x_4^2 + x_1 x_3^2 x_4^2 + x_1^2 x_2^2 x_4 + x_1^2 x_2 x_3^2) = 0$
e	5	$C_5^2 \ltimes C_4$	$4 \cdot 5 \cdot 5 = 100$	832	0	$\sum_{1 \le i \le 4} x_1^5 = 0$
f	4	1	$1 \cdot 1 \cdot 1 = 1$	208	8	Generic
g	4	D_8	$2 \cdot 2 \cdot 2 = 8$	832	2	$\sum_{1 \le i \le 3} x_i^4 + a x_1^2 x_3^2 + y_1 y_3 = 0 = c x_1 x_3 (x_1^2 + x_3^2) + y_1^2 + y_3^2$

Table 7. Effective examples.

the preimages of the node in the normalization. The possibilities, as listed in table 15, are $r \in \{2,4,7,8\}$.

The estimate for $|G|$ is then $[K : K'] \cdot |K'| \cdot |H'| \leq 2 \cdot 8 \cdot 40 = 640$.

2.3.2.4 **When the torsion is even.** In this section we will consider only Godeaux surfaces with an even torsion (in particular we care about having a divisor of order 2). The estimate can get significantly better in this case. The relevant setup is when the Godeaux surface has torsion C_2, since the automorphisms of the ones with torsion C_4 have been computed in the previous section.

Let η be a divisor of order 2 and define $D := K + \eta$. Then $h^0(D) = 1$, that is, we have only one effective representative in $|K + \eta|$. This gives a canonical element of $|2K_S|$, namely $2D$, which is non-reduced.

Since there is a non-reduced element of multiplicity 2 in $|2K_S|$, most possibilities for the base points cannot happen. Indeed, the base points must have even multiplicity, so only the cases $(2,2)$ and (4) survives (the latter with smooth or singular generic member).

The important observation is that in this case there is a special point in the base of the fibration, the one with fiber containing $2D$ (it will contain also two exceptional curve in the case $(2,2)$ and three in the case (4) smooth). This point must be fixed by the whole action of G, hence the group acting on the base cannot be dihedral, but it must be cyclic.

Moreover, we can use the knowledge on the singularity of $2D$ to provide a better estimate on the number of singular fiber of the fibration.

Case $(2,2)$. First, we consider G_p, the subgroup of index at most 2 of G fixing the two base points. On T_pS there are two special directions (that cannot be exchanged): the direction of $2D$ and the one of the generic member of $|2K_S|$. Having at least one fixed point, H_p is cyclic. Looking at the fibration, we see that the fiber containing $2D$ has also two exceptional curves, hence its χ_{top} is at least $-2+1+1 = 0$. This reduces the number of singular fibers to at most $27 - 6 + 1 = 22$. On the other hand, if $H_p \cong C_m$, then there are at least $m+1$ singular fibers, from which we have $m + 1 \leq s \leq 22$ or $m \leq 20$ (since $3 \mid 21$). Summing up, $\left|H_p\right|$ is at most C_{20}.

On the other side, K_p is cyclic by Cartan's Lemma, and acts on the general curve of $|2K_S|$ that is smooth of genus 4, fixing two points. As before, results are listed in table 13 and we have that K_p is at most C_{16}.

The estimates is then $|G| \leq [G : G_p] \cdot |K_p| \cdot |H_p| \leq 2 \cdot 16 \cdot 20 = 640$. Note that this is also the case of the Godeaux surfaces with torsion C_4 and $\mathrm{Aut}(S) \approx D_8$, where $|K_p| = |H_p| = 2$.

Case (4)*, smooth generic member.* Again, we have two fixed directions on T_pS that cannot be exchanged, hence H is cyclic. In the fibration, the fiber containing $2D$ has 3 more exceptional curves, hence $\chi_{\text{top}} \geq 1$, and the number of singular fiber is then $s \leq 27 - 7 + 1 = 21$. If $H \cong C_m$, there are at least $m + 1$ singular fibers, hence $m + 1 \leq s \leq 21$ or $m \leq 20$.

The estimate for K is the same as before, topping at C_{16}. The total estimate is then $|G| \leq |K| \cdot |H| \leq 16 \cdot 20 = 320$.

Case (4)*, singular generic member.* As usual, H is cyclic, because the two directions of the special curves cannot be exchanged (since one is $2D$); there is at least one double fiber, hence the number of singular fibers is $s \leq 20 - 1 = 19$ because double fibers have a discrepancy of at least 2 in the formula for the Euler characteristic (recall that this is a fibration in curves of genus 3, that needs only one blow up of p). If $H \cong C_m$, there are at least $m + 1$ singular fibers, hence $m + 1 \leq s \leq 19$ or $m \leq 18$. That is, H is at most C_{17}.

We need to estimate K, but if we look at the subgroup of G acting trivially on $\mathbb{P}T_pS$ we are not looking at K but at a subgroup of index 1 or 2: the one acting trivially on the fibration and not exchanging the tangent directions of the generic member of $|2K_S|$ at p. By Cartan's Lemma, K' is cyclic and acts on smooth curves of genus 3, fixing two points. The maximum order attainable is 8.

The estimates is then $|G| \leq [K : K'] \cdot |K'| \cdot |H| \leq 2 \cdot 8 \cdot 17 = 272$.

All results obtained are collected in table 8. They are split by torsion (even or odd), smoothness of the general member of $|2K_s|$, by the number and multiplicity of the base points, and finally by admitting, or not, that $2K_S$ can induce an isotrivial fibration.

2.3.3 *Actions of cyclic groups on a curve*

We collected in tables 9–14 the possible faithful actions of a cyclic group of order r on a smooth curve of genus g. The different tables corresponds to different requirements on the stabilizers of the action.

E	Gen. memb.	B. p.	Adm. isotr.?	max $\|G\|$	Bigg. data
No	Smooth	$(1,1,1,1)$	Yes/no	1600	100
		$(2,1,1)$	Yes/no	260	—
		$(3,1)$	Yes	416	—
			No	260	—
		$(2,2)$	Yes	832	—
			No	520	—
		(4)	Yes	416	—
			No	260	—
	Singular	(4)	Yes/no	640	—
Yes	Smooth	$(2,2)$	Yes	320	8
			No	640	8
		(4)	Yes	320	—
			No	200	—
	Singular	(4)	Yes/no	272	—

TABLE 8. Best estimates in various subcases of no fixed components in $|2K_S|$. E stands for even torsion.

r	g_C	g_C	Q	Additional ramification
2	$(6-Q)/4$	1	2	$(\bullet)^2$
		0	6	$(\bullet)^6$
3	$(4-Q)/6$	0	4	$(\bullet)^2$
4	$(2-Q)/8$	0	2	$(\bullet\bullet)$
5	$(0-Q)/10$	0	2	

TABLE 9. Cyclic groups acting on F with ramification containing $(\bullet)^4$.

r	g_C	g_C	Q	Additional ramification
2	$(8-Q)/4$	2	0	
		1	4	$(\bullet)^4$
		0	8	$(\bullet)^8$
4	$(6-Q)/8$	0	4	$(\bullet\bullet)^3$
				$(\bullet)^2$
6	$(4-Q)/12$	0	4	$(\bullet\bullet)$
10	$(0-Q)/20$	0	0	

Table 10. Cyclic groups acting on F with ramification containing $(\bullet)^2(\bullet\bullet)$.

r	g_C	g_C	Q	Additional ramification
2	$(7-Q)/4$	1	3	$(\bullet)^3$
		0	7	$(\bullet)^7$
3	$(6-Q)/6$	1	0	
		0	6	$(\bullet)^3$
4	$(5-Q)/8$	0	5	$(\bullet),(\bullet\bullet)$
5	$(4-Q)/10$	0	4	$(\bullet)$
6	$(3-Q)/12$	0	3	$(\bullet\bullet\bullet)$
9	$(0-Q)/18$	0	0	

Table 11. Cyclic groups acting on F with ramification containing $(\bullet)^3$.

r	g_C	g_C	Q	Additional ramification
2	$(9-Q)/4$	2	1	$(\bullet)$
		1	5	$(\bullet)^5$
		0	9	$(\bullet)^9$
4	$(9-Q)/8$	0	9	$(\bullet\bullet)^3(\bullet)$ $(\bullet)^3$
6	$(9-Q)/12$	0	9	$(\bullet\bullet\bullet)^3$ $(\bullet\bullet)(\bullet)$
10	$(9-Q)/20$	0	9	$(\bullet)$
12	$(9-Q)/24$	0	9	$(\bullet\bullet\bullet)$
18	$(9-Q)/36$	0	9	$(\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$

Table 12. Cyclic groups acting on F with ramification containing $(\bullet)(\bullet\bullet)$.

r	g_C	g_C	Q	Additional ramification
2	$(8-Q)/4$	2	0	
		1	4	$(\bullet)^4$
		0	8	$(\bullet)^8$
3	$(8-Q)/6$	1	2	$(\bullet)$
		0	8	$(\bullet)^4$
4	$(8-Q)/8$	1	0	
		0	8	$(\bullet)^2, (\bullet\bullet)$ $(\bullet\bullet)^4$
5	$(8-Q)/10$	0	8	$(\bullet)^2$
6	$(8-Q)/12$	0	8	$(\bullet\bullet)^2$
8	$(8-Q)/16$	0	8	$(\bullet\bullet\bullet\bullet\bullet)^2$
9	$(8-Q)/18$	0	8	$(\bullet)$
10	$(8-Q)/20$	0	8	$(\bullet\bullet)$
12	$(8-Q)/24$	0	8	$(\bullet\bullet\bullet\bullet\bullet)$
16	$(8-Q)/32$	0	8	$(\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$

Table 13. Cyclic groups acting on F with ramification containing $(\bullet)^2$.

r	g_C	g_C	Q	Additional ramification
2	$(9-Q)/4$	2	1	$(\bullet)$
		1	5	$(\bullet)^5$
		0	9	$(\bullet)^9$
3	$(10-Q)/6$	1	4	$(\bullet)^2$
		0	10	$(\bullet)^5$
4	$(11-Q)/8$	1	3	$(\bullet)$
		0	11	$(\bullet)^3, (\bullet\bullet)$
				$(\bullet), (\bullet\bullet)^4$
5	$(12-Q)/10$	0	12	$(\bullet)^3$
6	$(13-Q)/12$	0	13	$(\bullet)^2, (\bullet\bullet\bullet)^2$
				$(\bullet), (\bullet\bullet)^2$
				$(\bullet\bullet), (\bullet\bullet\bullet)^3$
8	$(15-Q)/16$	0	15	$(\bullet), (\bullet\bullet\bullet\bullet)^2$
9	$(16-Q)/18$	0	16	$(\bullet)^2$
10	$(17-Q)/20$	0	18	$(\bullet), (\bullet\bullet)$
12	$(19-Q)/24$	0	19	$(\bullet), (\bullet\bullet\bullet\bullet)$
				$(\bullet\bullet), (\bullet\bullet\bullet)$
14	$(21-Q)/28$	0	21	$(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)^2$
15	$(22-Q)/30$	0	22	$(\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet\bullet)$
16	$(23-Q)/32$	0	23	$(\bullet), (\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$
18	$(25-Q)/32$	0	25	$(\bullet\bullet), (\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$

Table 14. Cyclic groups acting on F with ramification containing $(\bullet)$.

r	g_C	g_C	Q	Additional ramification
2	$(6-Q)/4$	1	2	$(\bullet)^2$
		0	6	$(\bullet)^6$
3	$(6-Q)/6$	1	0	
		0	6	$(\bullet)^3$
4	$(6-Q)/6$	0	6	$(\bullet\bullet)^3$ $(\bullet)^2$
6	$(6-Q)/12$	0	6	$(\bullet\bullet\bullet)^2$
7	$(6-Q)/14$	0	6	$(\bullet)$
8	$(6-Q)/16$	0	6	$(\bullet\bullet)$
9	$(6-Q)/18$	0	6	$(\bullet\bullet\bullet)$
12	$(6-Q)/24$	0	6	$(\bullet\bullet\bullet\bullet\bullet\bullet)$

Table 15. Cyclic groups acting on F of genus 3 with ramification containing $(\bullet)^2$.

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