Factoring Heron

Vaughan Pratt

Vaughan Pratt (pratt@cs.stanford.edu) was born in Melbourne, Australia, took his undergraduate degree in pure mathematics and physics at the University of Sydney, and obtained his Ph.D. in computer science at Stanford University in 1972 under Donald Knuth. He taught at MIT from 1972 to 1981 and at Stanford from 1981 to 2000, where he is now professor emeritus. His research interests include logic, computing, combinatorics, geometry, and physics. Other pursuits are singing, philosophy, climatology, and politics. He is married with two daughters.

The pleasing symmetry of Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle with sides a, b, c and semiperimeter $s = (a+b+c)/2$ has attracted a considerable literature, most focusing just on inferring the formula from the Pythagorean theorem or other geometric facts [**1–11**]. This note gives a new proof of the well-known equivalence of Heron's formula and the Pythagorean theorem by applying Al-Karkhi's rule to the factorization $s(s-a) \times (s-b)(s-c)$.

Theorem. *Pythagoras if and only if Heron.*

Proof. By the "quarter squares" rule $4pq = (p+q)^2 - (p-q)^2$ of Al-Karkhi,

$$4s(s-a)(s-b)(s-c) = (s(s-a) + (s-b)(s-c))^2$$
$$- (s(s-a) - (s-b)(s-c))^2.$$

The sum $s(s-a) + (s-b)(s-c)$ simplifies to bc while the difference $s(s-a) - (s-b)(s-c)$ simplifies to $(b^2 + c^2 - a^2)/2$.

(If) If b and c are perpendicular, $bc = 2A$, so by Heron $a^2 = b^2 + c^2$.

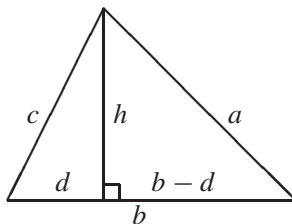(Only if) Erect h perpendicular to base b as shown in Figure 1.

Figure 1. Triangle with area $A = bh/2$.

Substituting $d^2 + h^2$ for c^2 and $(b-d)^2 + h^2$ for a^2 in $(b^2 + c^2 - a^2)/2$ reduces it to bd. Hence $4s(s-a)(s-b)(s-c)$ reduces to $(bc)^2 - (bd)^2$ or $b^2(c^2 - d^2)$, which by Pythagoras is b^2h^2 or $4A^2$. ■

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> The Pythagorean Theorem is regarded as the most fascinating Theorem of all of Euclid, so much so, that thinkers from all classes and nationalities, from the aged philosopher in his armchair to the young soldier in the trenches next to no-man's-land, 1917, have whiled away hours seeking a new proof of its truth.
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> —Elisha Scott Loomis, *The Pythagorean Proposition*

> The strict renunciation of eating meat follows almost as inevitably from the doctrine of the transmigration of souls as it can be derived from the oldest reports about Pythagoras. If the soul can enter into the bodies of animals, then the taboo on killing humans must be extended to them. Otherwise there would be a danger of killing not only another member of one's own species, but also possibly even a close relative, as Empedocles drastically puts it.
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> —Christoph Riedweg, *Pythagoras: His Life, Teaching, and Influence*, p. 68