Optimal Mechanism Design
(without Priors)

Jason D. Hartline
Microsoft Research – Silicon Valley

June 5, 2005

Also at EC

Sunday 2:00: G. Aggarwal, J. Hartline,
Knapsack Auctions.

Sunday 2:30: M.-F. Balcan, A. Blum, J. Hartline, Y. Mansour,
Sponsored Search Auction Design via Machine Learning.

Monday 8:55: M. Saks, L. Yu,
Weak monotonicity suffices for truthfulness on convex domains.

Monday 3:30: A. Ronen, D. Lehmann,
Nearly Optimal Multi Attribute Auctions.

Monday 3:55: E. David, A. Rogers, N. Jennings, J. Schiff, S. Kraus,
Optimal Design of English Auctions with Discrete Bid Levels.

Monday 4:45: M. Hajiaghayi, R. Kleinberg, M. Mahdian, D. Parkes,
Online Auctions with Re-usable Goods.

Tuesday 8:55: R. McGrew, J. Hartline,
From Optimal Limited to Unlimited Supply Auctions.

Tuesday 9:45: C. Borgs, J. Chayes, N. Immorlica, M. Mahdian, A. Saberi,
Multi-unit auctions with budget-constrained bidders.

Optimal Mechanism Design

Basic Question: how should a resource provider service consumers to maximize profit?

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Priors: known distributional information on consumer preferences.

Outline

Part I: Optimal Mechanism Design with Priors.
(game theory basics, truthful characterization, Myerson's optimal mechanism)

Part II: The Market Analysis Metaphor.
(emperical distributions, consistency issues, random sampling, machine learning, pricing algorithms)

Part III: Optimal Mechanism Design in Worst-case.
(competitive analysis, lower bounds, upper bounds, reduction to decision problem)

Part IV: Removal of Standard Assumptions.
(online auctions, collusion, asymmetric auctions, asymmetric settings)

Optimal Mechanism Design without Priors

Part I

Optimal Mechanism Design with Priors

Example Problem: Single-item Auction

Setting:

- Seller with one item.
- Bidders with *private valuations*: $v_1, \ldots, v_n$.

Design Goal:

- Single-round auction: bidders submit bids, seller decides winner and price.
- Truthful auction: bidders have incentive to bid true values.
- Optimal auction: seller gets optimal profit.

Economics Approach

Economics Approach to profit maximization:

1. Assume bidders’ valuations are random.
2. Characterize class of truthful mechanisms.
3. Find optimal mechanism from class for distribution.

Step 1: Valuations are Random

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The Independent Private Value (IPV) model:

1. Bidder i has valuation $v_i \in [0, h]$ distributed as F_i.

   Cumulative distribution function: $F_i(b) = \textbf{Pr}[v_i \geq b]$.
   Probability density function: $f_i(b) = F_i'(b)$.

2. Bidder's values are independent:

   Joint density function: $f(\mathbf{b}) = \prod_i f_i(b_i)$

Definition: f is the *prior distribution*, known to seller.

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Example:

- Input: $\mathbf{b} = (1, 3, 6, 2, 4)$.
- Output: the 6 bid wins and pays 4.

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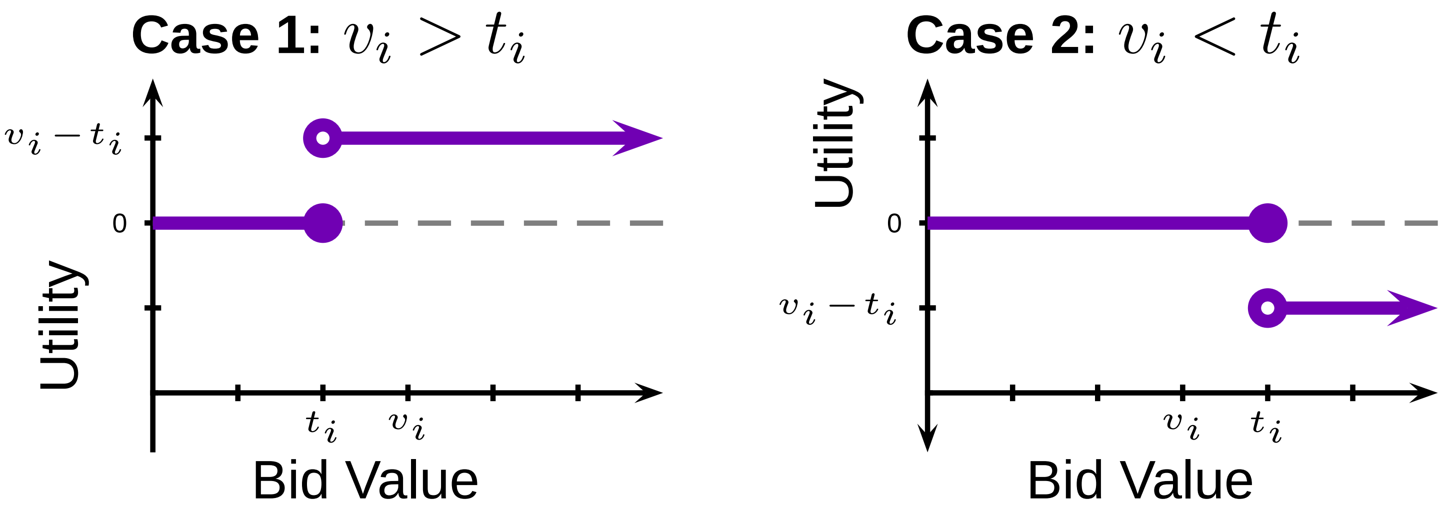

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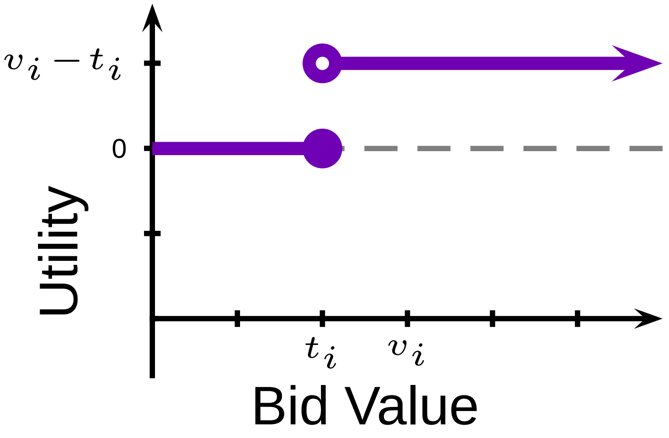

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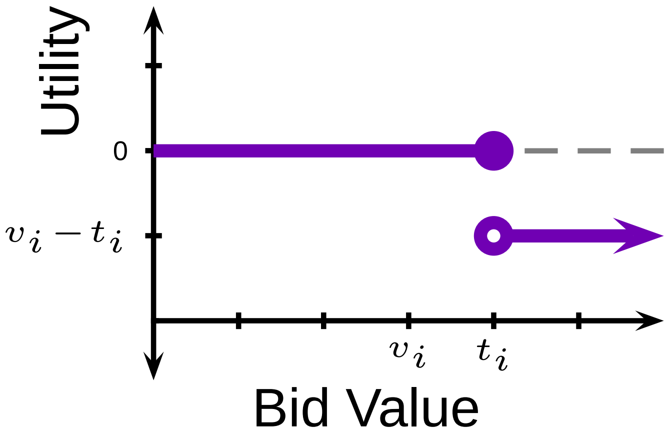

Result: In either case, bidder i's best strategy is to bid $b_i = v_i$!

Bid-Independence

Definition: Bids with bidder i removed:

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Bid-Independent Auction: BI_g

On input $\mathbf{b}$, for each bidder i:

1. $t_i \leftarrow g(\mathbf{b}_{-i})$.
2. If $t_i < b_i$, sell to bidder i at price t_i.
3. If $t_i > b_i$, reject bidder i.

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Theorem: A (deterministic) auction is truthful iff it is bid-independent.

Notational Interlude

Notation: for input, $\mathbf{b}$,

- $\mathbf{x} = (x_1, \ldots, x_n)$: x_i is indicator for bidder i getting the item.
- $\mathbf{p} = (p_1, \ldots, p_n)$: p_i is bidder i's payment .
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Note: Output of mechanism, $(\mathbf{x}, \mathbf{p})$, is function of $\mathbf{b}$.

- Explicitly: $\mathbf{x}(\mathbf{b}), x_i(\mathbf{b}), x_i(b_i, \mathbf{b}_{\text{-}i})$, and $\mathbf{p}(\mathbf{b})$, etc.
- With $\mathbf{b}_{\text{-}i}$ implicit: $x_i(b_i)$ and $p_i(b_i)$.

Step 3: Find Optimal Mechanism

Step 3: Find Optimal Mechanism from class for distribution.

Maximize Auction's Profit: $\mathbf{E}_{\mathbf{b}}[\sum_i p_i(\mathbf{b}) - c(\mathbf{x}(\mathbf{b}))]$.

Subject to truthfulness:

1. bidder i wins if $b_i > t_i \Leftrightarrow x_i(b_i)$ is a step function.
2. bidder i pays $t_i x_i(b_i) \Leftrightarrow p_i(b_i) = x_i(b_i)b_i - \int_0^{b_i} x_i(b)db$.

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Definition: The *virtual valuation* of a bidder i with value $v_i \sim F_i$ is

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

Lemma: For $x_i(\mathbf{b})$ and bids $\mathbf{b}$ with joint densify function f:

$$\mathbf{E}_{\mathbf{b}}[p_i(\mathbf{b})] = \int_{\mathbf{b}} \psi_i(b_i) x_i(\mathbf{b}) f(\mathbf{b}) d\mathbf{b}.$$

Proof of Lemma

$$
\begin{aligned}
\mathsf{E}_{\mathbf{b}}\left[p_i(\mathbf{b})\right] &= \int_{\mathbf{b}} p_i(b_i) f(\mathbf{b})\, d\mathbf{b} \\
&= \int_{\mathbf{b}_{-i}} \int_{b_i} p_i(b_i) f_i(b_i) f(\mathbf{b}_{-i})\, db_i\, d\mathbf{b}_{-i} \\
&= \int_{\mathbf{b}_{-i}} \int_{b_i} \left[x_i(b_i) b_i - \int_0^{b_i} x_i(b)\, db \right] f_i(b_i)(\mathbf{b}_{-i})\, db_i\, d\mathbf{b}_{-i} \\
&= \int_{\mathbf{b}_{-i}} \left[\int_{b_i} x_i(b_i) b_i f_i(b_i)\, db_i - \int_{b_i=0}^{h} \int_{b=0}^{b_i} x_i(b) f_i(b_i)\, db\, db_i \right] f(\mathbf{b}_{-i})\, d\mathbf{b}_{-i} \\
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\begin{aligned}
\mathsf{E}_{\mathbf{b}}\left[p_i(\mathbf{b})\right] &= \int_{\mathbf{b}} p_i(b_i) f(\mathbf{b})\, d\mathbf{b} \\
&= \int_{\mathbf{b}_{-i}} \int_{b_i} p_i(b_i) f_i(b_i) f(\mathbf{b}_{-i})\, db_i\, d\mathbf{b}_{-i} \\
&= \int_{\mathbf{b}_{-i}} \int_{b_i} \left[x_i(b_i) b_i - \int_0^{b_i} x_i(b)\, db\right] f_i(b_i)(\mathbf{b}_{-i})\, db_i\, d\mathbf{b}_{-i} \\
&= \int_{\mathbf{b}_{-i}} \left[\int_{b_i} x_i(b_i) b_i f_i(b_i)\, db_i - \int_{b_i=0}^{h} \int_{b=0}^{b_i} x_i(b) f_i(b_i)\, db\, db_i\right] f(\mathbf{b}_{-i})\, d\mathbf{b}_{-i} \\
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&= \int_{\mathbf{b}_{-i}} \int_{b_i} \left[b_i - \frac{1 - F_i(b_i)}{f_i(b_i)}\right] x_i(b_i) f_i(b_i) f(\mathbf{b}_{-i})\, db_i\, d\mathbf{b}_{-i} \\
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Myerson

Step 3: Find optimal mechanism.

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Theorem: [Mye-81] Given allocation rule $\mathbf{x}$ and bids $\mathbf{b}$ with density function f the expected profit is

$$\int_{\mathbf{b}} \left[\sum_i \psi_i(b_i) x_i(\mathbf{b}) - c(\mathbf{x}(\mathbf{b}))\right] f(\mathbf{b}) d\mathbf{b}.$$

Definition: *Myerson's optimal mechanism* for distribution $\mathbf{F} = F_1 \times \ldots \times F_n$, is *Myersion*$_{\mathbf{F}}(\mathbf{b})$ with

$$\mathbf{x}(\mathbf{b}) = \mathrm{argmax}_{\mathbf{x}'} \sum_i \psi_i(b_i) x_i' - c(\mathbf{x}').$$

Theorem: Myersion's mechanism is optimal and truthful when the $\psi_i(\cdot)$s are monotone.

Note 1: This applies to any cost function $c(\mathbf{x})$ (not just for single-item auction).

Myerson

Note 2: For some $c(\mathbf{x})$ non-monotone $\psi_i(\cdot)$ can be *ironed* to be monotone.

Myerson

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Example: Basic Auction

The Basic Auction Problem:

Given:

- n identical items for sale.
- n bidders, bidder i willing to pay at most v_i for an item.

Design: auction with maximal profit.

Example

Recall Theorem: [Mye-81] Given allocation rule $\mathbf{x}$ and bids $\mathbf{b}$ with density function f the expected profit is

$$\int_{\mathbf{b}} \left[\sum_i \psi_i(b_i)x_i(\mathbf{b}) - c(\mathbf{x}(\mathbf{b}))\right] f(\mathbf{b})d\mathbf{b}.$$

Recall Example: single-item auction

$$c(\mathbf{x}) = \begin{cases} 0 & \text{if } \sum_i x_i \leq 1 \\ \infty & \text{otherwise.} \end{cases}$$

Result:

- Winner: the bidder with highest $\psi_i(b_i)$ (such that $\psi_i(b_i) \geq 0$).
- Winner's Payment: $\mathrm{argmin}_b\{\psi_i(b) \geq \psi_j(b_j)\ \&\ \psi_i(b) \geq 0\}$

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- Suppose bids are identical, $F_i = F_j$:
  $\Rightarrow \max\{b_j \ : \ j \neq i\} \cup \{\psi^{-1}(0)\}$

Example

- Interpretation: Optimal Auction = Vickrey w/reserve price $\psi^{-1}(0)$.

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Definition: $\mathrm{opt}(F) = \psi^{-1}(0) = \mathrm{argmax}_b\, b(1 - F(b))$

Other Directions

1. General ironing procedure for arbitrary costs?
2. Agent's with correlated values. [Ron-03].
3. Deficits. [CHRSU-04]
4. Iterative Mechanisms. [DRJSK-05]
5. Optimal Mechanism for multi-parameter agents?
   (needs characterization like [SW-05], related to [RL-05])

Optimal Mechanism Design without Priors

Part II

The Market Analysis Metaphor

Motivation

Where does known prior come from?

1. previous sales.
2. market analysis.

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1. incentive properties.
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Argument 1: by assuming a known prior we ignore incentive and performance issues from obtaining the prior.

Argument 2: (Wilson Doctrine) Mechanisms should be independent of details.

Market Analysis

Market Analysis Approach:

1. Market Analysis ⇒ distributional knowledge $\mathbf{F} = (F_1, \ldots, F_n)$
2. Design mechanism for $\mathbf{F}$: Myersion$_{\mathbf{F}}$

Recall Incentive Compatibility: for all i, $x_i(b_i)$ is monotone in b_i.

Can be arbitraty function of $\mathbf{b}_{\text{-}i}$!

Insight: use $\mathbf{b}_{\text{-}i}$ for market analysis.

Imperical Distributions

Definition: The *imperical distribution* for $\mathbf{b}$ is

$$\hat{F}_{\mathbf{b}}(x) = \frac{|\{i : b_i < x\}|}{n}.$$

Recall: Myersion$_F \Rightarrow x_i^F(\mathbf{b}), p_i^F(\mathbf{b})$

Set $x_i(b_i)$ be the allocation for bidder i in Myersion$_{\hat{F}_{\mathbf{b}_{-i}}}$

Estimating Distributions

Recall: Myerson's Optimal Auction for bids i.i.d. from F:

1. optimal price $= \operatorname{argmax}_p p(1 - F(p))$.
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Definition: The empirical distribution $\mathbf{b}_{\text{-}i}$ is

$$\hat{F}_{\mathbf{b}_{\text{-}i}}(p) = \text{“number of bids less than } p\text{”} \times \frac{1}{n-1}.$$

Deterministic Optimal Price Auction

For basic auction problem:

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Deterministic Optimal Price Auction (DOP)
[GHW-01,BV-03,Seg-03]

On input $\mathbf{b}$, for each bidder i:

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Lemma: Worst-case profit is bad. [GHW-01]

Worst Case Analysis of DOP

Example: for DOP and $\mathbf{b} = (\underbrace{\overbrace{10, 10, \ldots, 10}^{\text{10 bidders}}, 1, 1, \ldots, 1}_{\text{100 bidders}})$

Profit: $10 \times$ $+\ 90 \times$

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Example: for DOP and $\mathbf{b} = (\underbrace{\overbrace{10, 10, \ldots, 10}^{\text{10 bidders}}, 1, 1, \ldots, 1}_{\text{100 bidders}})$

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Revenue from 10 bid

What does DOP do for $b_i = 10$?

$$\operatorname{opt}(\mathbf{b}_{\text{-}i}) = \underbrace{(\overbrace{10, \ldots, 10}^{\text{9 bidders}}, 1, 1, \ldots, 1)}_{\text{99 bidders}}$$

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Result: Bidder i buys item at price 1!

Is $\mathrm{opt}(\mathbf{b}_{\text{-}i}) = 1$ or 10?

- Revenue$_{10}$ $= 10 \times 9 = 90$.
- Revenue$_{1}$ $= 1 \times 99 = 99$.
- Thus, $\mathrm{opt}(\mathbf{b}_{\text{-}i}) = 1$.

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Revenue from 1 bid

What does DOP do for $b_i = 1$?

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- Revenue$_{10}$ $= 10 \times 10 = 100.$
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Profit: $10 \times \underbrace{\text{Revenue from 10 bid}}_{1} + 90 \times \underbrace{\text{Revenue from 1 bid}}_{0} = 10$

Revenue from 1 bid

What does DOP do for $b_i = 1$?

$$\mathrm{opt}(\mathbf{b}_{-i}) = \underbrace{(\overbrace{10, 10, \ldots, 10}^{\text{10 bidders}}, 1, \ldots, 1)}_{\text{99 bidders}}$$

Result: Bidder i is rejected!

Is $\mathrm{opt}(\mathbf{b}_{-i}) = 1$ or 10?

- Revenue$_{10}$ $= 10 \times 10 = 100.$
- Revenue$_{1}$ $= 1 \times 99 = 99.$
- Thus, $\mathrm{opt}(\mathbf{b}_{-i}) = 10$.

General Consistency Issue

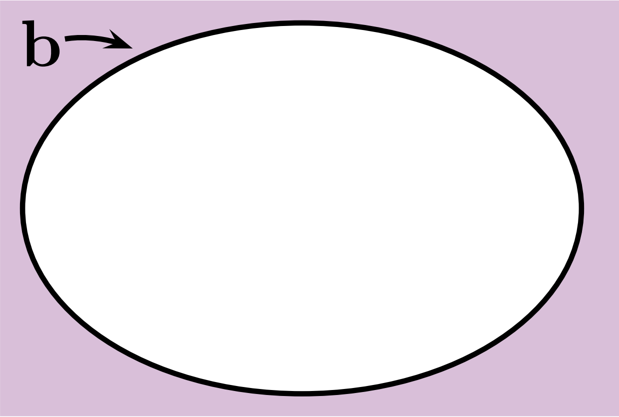
Emperical Myerson Auction may be inconsistent

Double Auction Problem.

Approximation via Random Sampling

Random Sampling Optimal Price Auction, RSOP

1. Randomly partition bids into two sets: $\mathbf{b}'$ and $\mathbf{b}''$.
2. Use $p' = \mathrm{opt}(\mathbf{b}')$ as price for $\mathbf{b}''$.



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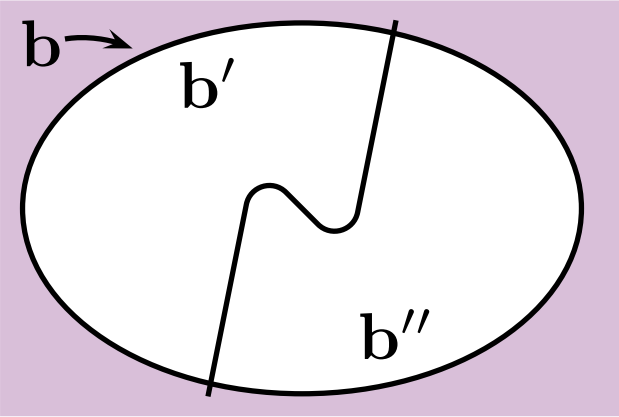

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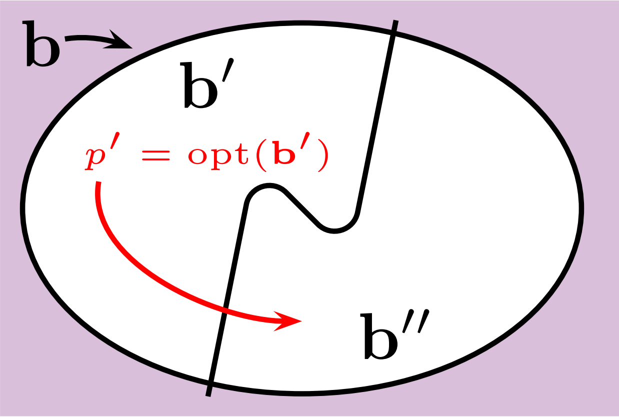

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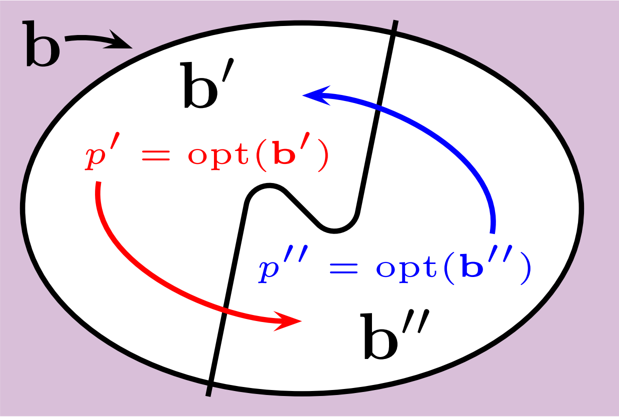

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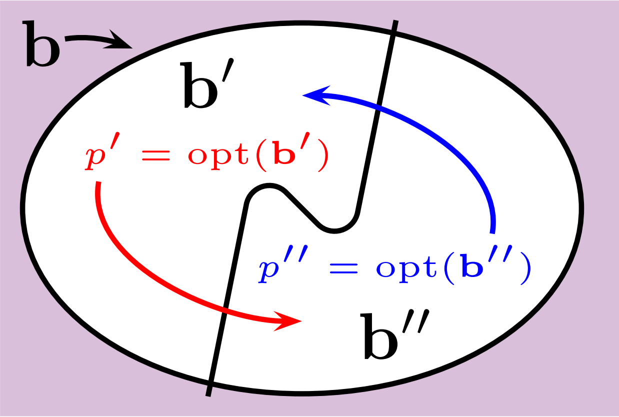

Theorem: For $\mathbf{b}$ on range $[1, h]$, profit of RSOP approaches optimal profit as $n \to \infty$.

Worst Case with Assumption

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Implicit Definition: optimal profit = “optimal profit from single price sale with bidders’ valuations.”

Worst Case with Assumption

Recall Theorem: For $\mathbf{b}$ on range $[1, h]$, profit of RSOP approaches optimal profit as $n \to \infty$.

Implicit Assumption: optimal profit $\gg h$.

Implicit Definition: optimal profit = "optimal profit from single price sale with bidders' valuations."

Fact: impossible to approximate optimal profit when it is optimal to sell only one unit.
E.g., $\mathbf{b} = (1, 1, 1, 1, h, 1, 1)$

Consistency

Concern: lack of consistency?

(bidders offered optimal prices from different empirical distributions)

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Recall: Myerson-VCG Construction:

1. Compute each player's *virtual valuation*

$$\phi(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}.$$

2. Run VCG on virtual valuations.

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Generalized DOP Technique: for each bidder i,

1. Compute virtual valuations using $\hat{F}_{\mathbf{b}_{-i}}$.
2. Compute outcome of VCG on virtual valuations for bidder i.

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Different empirical distributions $\Rightarrow$ inconsistency.

The Double Auction Problem

The *Double Auction Problem*:

Given:

- n sellers, seller i willing to sell a unit for at least s_i.
- n buyers, buyer i willing to buy a unit for at most b_i.

Design: Double auction maximize profit of broker. [BV-03,DGHK-02]

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Generalized DOP $\Rightarrow$ inconsistent.
Generalized RSOP $\Rightarrow$ consistent.

Generalizing RSOP

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2. Compute virtual valuations for $\mathbf{b}'$ and $\mathbf{s}'$ using $\hat{F}_{\mathbf{b}''}$ and $\hat{F}_{\mathbf{s}''}$.
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Subtlety: Must *iron* emperical distribution when it fails the *monotone hazard rate* condition.

Is consistency feasible?

Difficulty: Consistency, Truthfulness, and Profit Maximization.

Example:

- Basic Auction problem (n bidders, n units).
- *Envy-freedom:* all bidders are offered the same price.

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Theorem: Exists approximately optimal auctions that are

- truthful with high probability and envy-free, or
- envy-free with high probability and truthful.

Optimal Mechanism Design without Priors

Part III

The Worst Case

Analysis Framework

Recall Goal: Truthful profit maximizing basic auction.

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What is optimal public value auction?

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Optimal Single-Price Mechanism with Two Winners: $\mathcal{F}^{(2)}$

1. Compute best single sale price, p, for two or more items.
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Example:

- Input: $\mathbf{b} = (200, 11, 10, 2, 1)$.
- Output: the 200, 11, and 10 bids win at price 10.
- Revenue: 30.

Worst Case Competitive Auctions

Definition: A randomized auction is β-*competitive* in worst case if its expected profit is at least $\mathcal{F}^{(2)}/\beta$ for any input.

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2. 3.39-competitive randomized auction. [Goldberg, Hartline 2003]
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Main Theorem: No auction is better than 2.42-competitive.

Classical Reduction

Optimization problem: "What is the maximum value of a feasible sol ution?"

Decision problem: "Is there a feasible solution with value at least V?"

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Note: This reduction does not work for private value problems. (Simulating several truthful mechanisms and using the outcome of the best one is not truthful)

Basic Auction Decision Problem

The Decision Problem for the Basic Auction:

Given:

- n identical items for sale.
- n bidders, bidder i willing to pay at most v_i for an item.
- Target profit R.

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Definition: *Profit extractor* is solution to private value decision problem.

Moulin-Shenker

Result: [Moulin, Shenker 1996] Profit extractor for basic auction.

ProfitExtract_R

1. Find largest k s.t. k bidders have $b_i \geq R/k$.
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2. Bound **E**$\left[\mathcal{F}^{(2)}(\mathbf{B})\right]$.

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Recall: **E**$[\mathcal{A}(\mathbf{B})] = 2$, therefore competitive ratio is 2.

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Recall:

- For $\mathbf{b} = (b_1, b_2)$, $\mathcal{F}^{(2)}(\mathbf{b}) = 2\min(b_1, b_2)$.
- Vickrey Revenue $= \min(b_1, b_2)$.

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What is known:

- 2.3-competitive auction (note: $13/6 \approx 2.166$).
- Optimal auction uses prices $\neq$ bid values.
  (for prices $=$ bid values, optimal auction is 2.5-competitive)

General Lower Bound

Theorem: The competitive ratio of any auction is at least

$$1 - \sum_{i=2}^{n} \left(\frac{-1}{n}\right)^{i-1} \frac{i}{i-1} \binom{n-1}{i-1} \geq 2.42.$$

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Proof Outline:

1. Compute **E**$\left[\mathcal{F}^{(2)}(\mathbf{B})\right]$.
    (a) Compute **Pr**$\left[\mathcal{F}^{(2)}(\mathbf{B})\right] \geq z$.
    (b) Integrate.
2. Divide by **E**$[\mathcal{A}(\mathbf{B})] = n$.
3. Take limit.

Compute **Pr**$\left[\mathcal{F}^{(2)}(\mathbf{B}) \geq z\right]$

Lemma: **Pr**$\left[\mathcal{F}^{(2)}(\mathbf{B}) \geq z\right] = n\sum_{i=2}^{n}\left(\frac{-1}{z}\right)^{i} i\binom{n-1}{i-1}$.

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Proof: (high level)

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6. **Pr**$\left[\mathcal{F}^{(2)}(\mathbf{b}^{(n)}) > z\right] =$ **Pr**$[F_{n,0} > z] -$ **Pr**$[\mathcal{H}_1]$.

Conclusions

General:

- Upper Bound: 3.25. [HM-05]
- Lower Bound: 2.42. [GHKS-04]
- **Open:** optimal auction?

Limited Supply:

- 2-items: optimal competitive ratio = 2. [FGHK-02]
- 3-items: optimal competitive ratio = $13/6 \approx 2.17$. [GHKS-04,HM-05]
- 4-items: lower bound: $215/96 \approx 2.24$. [GHKS-04]

Optimal Mechanism Design without Priors

Part IV

The Technique of Consensus Estimates

Models

Analysis Models:

- Average Case.
- Worst Case.
  - Approximation with assumption.
  - Competitive analysis.

Design Techniques:

- Market analysis metaphor.
- Other techniques.

Incentive Properties:

- Truthful.
- Truthful with high probability.

Solution Approach

Consider definitions:

- A *summary value* does not change much when any bidder lowers their bids.

  E.g., $\#_p(\mathbf{b}) =$ "number of bidders above p"
  $\mathrm{OPT}(\mathbf{b}) =$ "optimal profit from a single price"

- A *summary consensus estimate* is a random estimate of summary value that with high probability cannot be manipulated by a bidder lowering their bid.

- A *summary mechanism*, $\mathcal{M}_{S_1,\ldots,S_k}$ is a consistent mechanism that approximates profit when parameterized by (an) approximate summary value(s).

Classical Reduction

Optimization problem: “What is the maximum value of a feasible solution?”

Decision problem: “Is there a feasible solution with value at least V?”

Classical reduction: Search for optimal value using repeated calls to decision problem solution.

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Note: This reduction does not work for private value problems.
(Simulating several truthful mechanisms and using the outcome of the best one is not truthful)

Basic Auction Decision Problem

The Decision Problem for the Basic Auction:

Given:

- n identical items for sale.
- n bidders, bidder i willing to pay at most v_i for an item.
- Target profit R.

Design: auction mechanism that obtains profit R if $R \leq \mathrm{OPT}$.

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Definition: *Profit extractor* is solution to private value decision problem.

Moulin-Shenker

Result: [Moulin, Shenker 1996] Profit extractor for basic auction.

$\mathrm{ProfitExtract}_R$

1. Find largest k s.t. k bidders have $b_i \geq R/k$.
2. Sell at price R/k.
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- $R = 9$.
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- *envy-free!*

Summary Consensus Estimates

Fact: If OPT sells at least k units,

$$\tfrac{k-1}{k}\,\mathrm{OPT}(\mathbf{b}) \;\leq\; \mathrm{OPT}(\mathbf{b}_{\text{-}i}) \;\leq\; \mathrm{OPT}(\mathbf{b})$$

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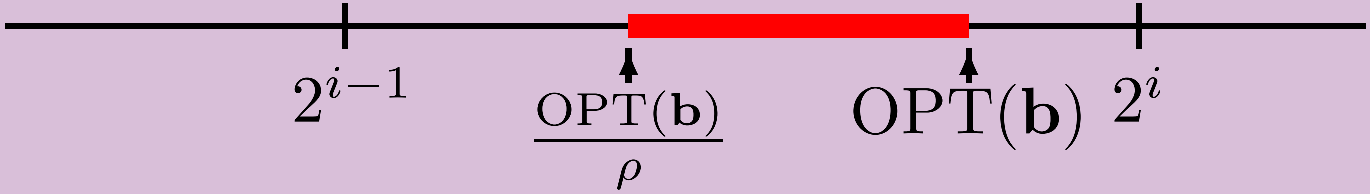

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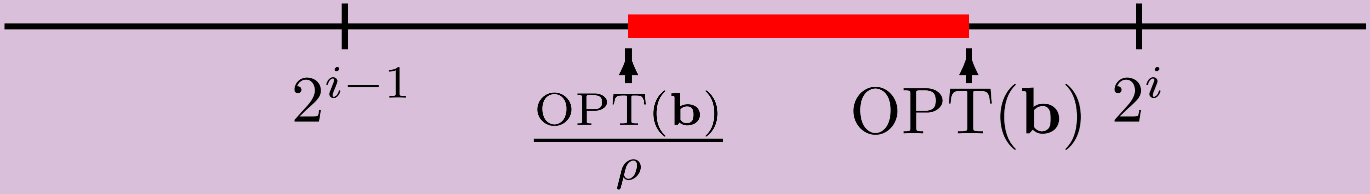

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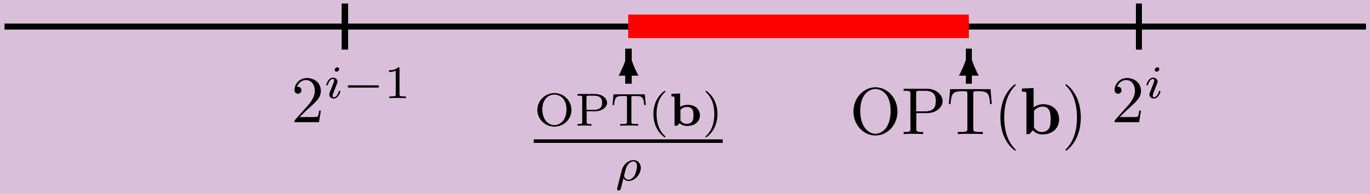

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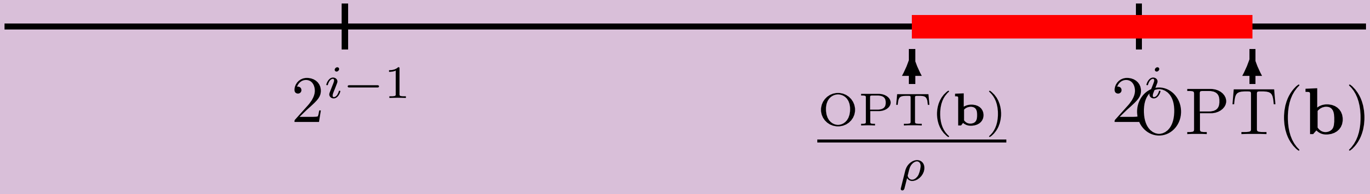

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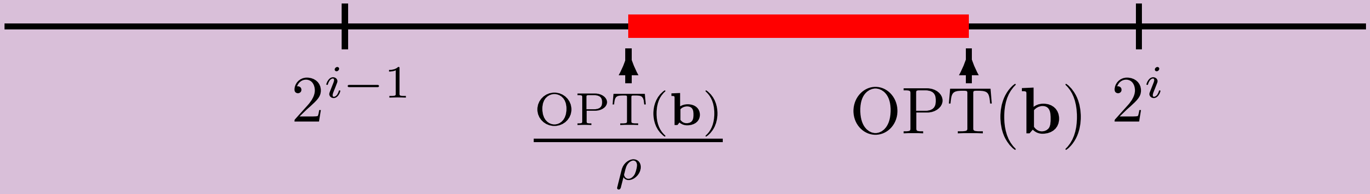
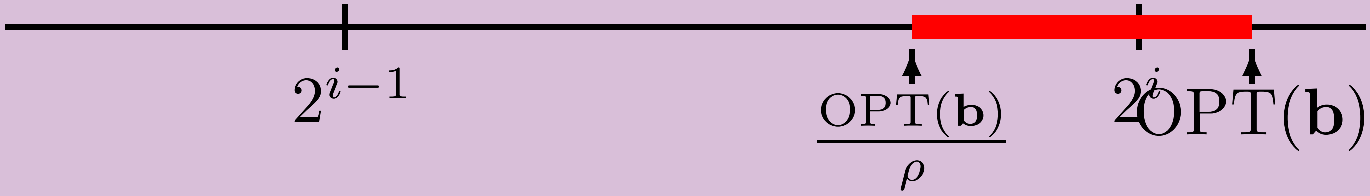


Case 2: No Consensus!



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$$\begin{aligned} 1 - \log \rho &= 1 + \log\left(1 - \tfrac{1}{k}\right) \\ &= 1 - O(1/k) \end{aligned}$$

Final Solution

Consensus and Profit Extraction Auction, CoPE

On input $\mathbf{b}$,

1. Draw y uniform $[0,1]$.
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From [GH-03]:

Theorem: CoPE auction is truthful with high probability.

Theorem: CoPE auction is envy-free.

Theorem: CoPE auction approximates the optimal profit.

Notes on CoPE

Motivates Search for Profit Extractors.

- Exists (approximate) profit extractor for double auciton.
- Exists profit extractor for decreasing marginal costs.
- **Open:** profit extractors for other constrained optimizations?

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  - Our Choice: optimal single price sale of at least two units.
  - Choise of $\mathcal{G}$ is mostly irrelevant. [HM-05]

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3. Future Directions:
   - Approximating general optimization problems.
     (with cost functions or constrained feasible outcomes)
   - Asymmetric optimizations.

Followup to Wilson

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