Constructing and Defining 3D Polyhedra: A Design Study Fostering Early Mathematical Practice and Visualization

Megan Wongkamalasai, Rich Lehrer, Vanderbilt University
megan.j.wongkamalasai@vanderbilt.edu , rich.lehrer@vanderbilt.edu

Abstract: Developments within STEM fields increasingly rely upon new tools and methods of visualization; yet, current early elementary mathematics curriculum largely neglect cultivating children's mathematical visualization skills. During a six day classroom design experiment, we studied how young children co-developed concepts of space and 3D shape and practices of defining and visualization. Our design is shaped by our commitments to theoretical developments in the learning science that conceptualize learning as based in participation within local mathematical practices that emerge from everyday activity.

Supporting early mathematical practice through space and visualization

Geometry and spatial mathematics are often overlooked in early mathematics education. When forms of spatial mathematics are taught, instruction is often constrained to the naming and the decomposition and composition of shapes (NRC, 2009). Yet, as developments in STEM fields increasingly rely upon visualization tools and spatial reasoning, there is an increasing attention to what forms of early geometry experience might be foundational for students' long term STEM learning and engagement (NRC, 2006). Recent work in early geometry has helped to identify forms of activities and tools that might support children's geometry learning (Hawes et al., 2017; Ng & Sinclair, 2015; Clements & Sarama, 2007). However, less is known about how geometry activities and tools operate within the learning ecology to support young students in co-developing mathematical concepts of space and mathematical practices. This is critical, as studies from the learning science have demonstrated that visualization is intricately tied to disciplinary epistemic practices or ways of knowing (Stevens & Hall, 1998; Goodwin, 1994). Thus, if the goal is to systematically develop students' visualization skills, these skills must develop in relation to the practices and goals of a discipline. In our instructional design experiment, we aimed to study how the learning ecology and forms of visualization support young children's (grade 1) co-development of the mathematical practice of defining and concepts relating to 3D polyhedra and shape (primarily pyramids and prisms).

Context

The context of this work is a six day instructional unit on defining properties and classes of 3D polyhedra. This is the first implementation of this instructional unit developed by the research team. We shared a written form of the unit with the focus teacher, Mrs. B; however, this document acted as a boundary object between the research team and teacher to help negotiate and guide classroom instruction and activities rather than as a set curriculum. Thus, the instructional design was emergent over the six days to account for developments in student thinking, history of activity, teacher input, and classroom sociomathematical norms. At the end of the unit, each student engaged in a one-on-one post-interview with the first author, M.W.

The study took place in first grade classroom in a school located in a Southeastern, rural town in the U.S. The classroom has 21 students with a range of cultural backgrounds (48% of students are emergent bilinguals). The classroom teacher, Mrs. B, has been teaching first grade for 10 years; however, this is the first time that she has engaged with this particular content area. Additionally, Mrs. B has a practice of eliciting student thinking and often asks for students to share their solutions, but she has not engaged students in defining. Thus, not only is the practice of defining new for students, but it is also new for Mrs. B in the context of mathematics. Despite these new defining goals, Mrs. B acted as the lead teacher throughout the instructional unit, and M.W. served as a co-instructor. In the capacity of co-instructor, M.W. talked with students during small group activities, posed questions to the class during whole group to redirect discussions, and helped Mrs. B make instructional decisions around what next steps would be most fruitful.

Instructional design and implementation

In this section, we first describe how our initial conjectures about what instructional supports and activities might help meet the goal of co-developing students' defining practice and concepts of shape. These conjectures informed the initial design of instruction, and they convey our theoretical commitments and assumptions. Next, we provide a summary of the enacted instructional design that highlights salient moments of student learning. We conclude with results from the post-interviews to provide a more individual look at students' development of defining and concepts of shape.

Design conjectures and theoretical perspectives

We build on sociocultural views of mathematics that position mathematics as situated in everyday activity and defined by the evolving practices and goals of the local community (Saxe & Esmonde, 2012; Lehrer & Lesh, 2003; Gravemeijer, 1999; Lave et al.,1984;). Thus, in designing instruction to support students' development of mathematical concepts and practices, we not only attend to the development of students' concepts, but also to how the activities and instructional support help foster a mathematical community defined by their evolving collective goals and practices. Because we are working with young children, we take particular care in considering how children's existing forms of activities and goal can serve as the foundations for continued mathematical learning and engagement. This consideration helped inform the activity context of the unit—3D construction and design.

We aimed to look at how 3D construction and design activities can be leveraged to develop children's concepts of 3D shape (e.g., properties and measures of 3D polyhedra) and defining and visualization practices. Not only is 3D construction an activity familiar to many children, but we conjectured that the goals children bring to this task might align with mathematical goals of defining. For example, children often build structures for self-expression, and they take great pride in describing their structures to others. The practice of defining in mathematics plays a similar communicative function in mathematics; however, unlike in construction play, defining in mathematics goes beyond self-description and instead requires negotiation around the words used to describe in order to establish shared meaning (Kobiela & Lehrer, 2015). We hoped that having children build and describe 3D structures to others would create a need for a shared language of description. In order to develop a language of description focusing on mathematical properties of shape, we chose to use construction materials that would likely elicit children's existing mathematical resources, Magformers or 2D magnetic polygons (see Figure 2).

Definition in mathematics also plays a critical epistemic role by helping establish relations and distinctions between concepts that can lead to new mathematical objects and questions for inquiry (Kobiela & Lehrer, 2015). In alignment of our goals to establish a practice of defining and concepts of 3D shape, we conjectured that students would need to do more than describe their 3D structures to each other in order to support defining. Thus, we conjectured that by having students compare their structures to teacher identified sets of structures, students would be more likely to attend to salient properties and relations between properties (i.e., number of faces in relation to the shape of the base of a pyramid).

Finally, mathematical conceptions of space require visualizing and attending to patterns of invariances and change through dynamic transformations (NCTM, 2014). A second goal of our study was to support children's mathematical visualization; thus, rather than considering 3D shapes as static, we emphasized defining properties of space dynamically. By considering properties of space in relation to motions and transformations (i.e., rotations) we hoped that this would support students in using visualization to further refine relations and properties of shape. In the unit reported here, students primarily used rotations to consider how their sense of different properties was or was not consistent with a new position in space. In our ongoing project, students develop further conceptions of transformations to consider properties in relation to rotation and reflection symmetries; however, we thought it important that students first develop a shared language to describe what properties of space are preserved by different transformations.

Implementation

Day 1-2: Student constructions and descriptions of their structures

Mrs. B began the unit by communicating to students the goal of the unit: to construct and describe 3D structures to others. To elicit some initial description words, Mrs. B presented the class with two different structures—a closed tetrahedron and a tetrahedron missing one face. She then asked students how they would describe these structures to others. Students generated a list of descriptive words that included both properties and name of 3D polyhedra and 2D polygons (Figure 2A). When students described a property of 3D shapes using everyday language and gesturing to the corresponding part on the tetrahedrons, Mrs. B would record the word used by the students and relate this to the mathematical word (i.e., sides are faces). Additionally, not all of the words produced were accurate. For example, students described the tetrahedrons as cones, pyramids, and triangular prisms. Mrs. B recorded all words with the hopes that clarification of these words would emerge over the course of the unit.

After developing a list of words that could be used to describe 3D structures, students used Magformers to build whatever 3D structures they liked, and as students built their structures, Mrs. B and M.W. circled the classroom asking students to describe their structures using the words previously produced. When students introduced a new descriptive word, such as corners, Mrs. B would pause the class and introduce the new word before adding it to the chart of descriptive words. Figure 1 shows a representative sample of the types of structures produced by the class. After students had time to explore what the different types of structures they could make with the Magformers, Mrs. B asked students to write down a description of one of their structures to share with the class, pointing to the list of descriptive words as a guide. While students wrote down their descriptions, Mrs. B and M.W. looked for student descriptions that ranged from broad (it is a closed 3D structure) to more specific (it is a pyramid with 4 triangle faces).

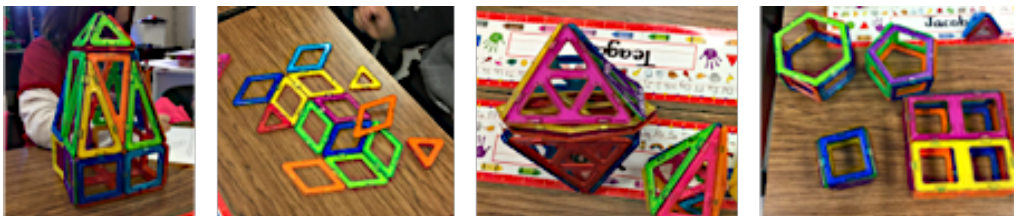
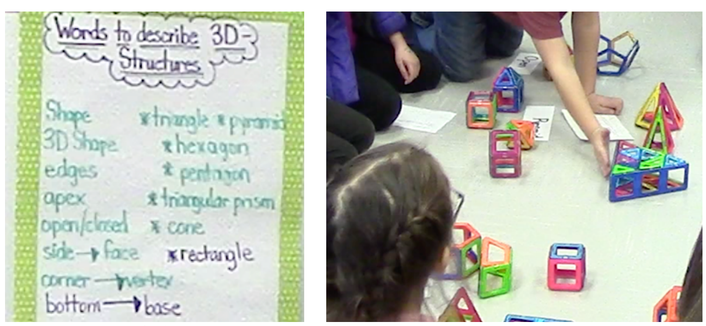


Figure 1: Examples of students' intial constructions with Magformers

Next, students brought their written descriptions and structures to the floor for a whole group discussion. Mrs. B selected students to read their descriptions of their structures, beginning with students with very broad descriptions. Our intention in starting with broad descriptions and moving towards more specific descriptions, was that students would begin to see the importance of precise descriptions in distinguishing between structures. As students read their descriptions, Mrs. B asked students to identify what descriptive words the student used from their list. She then wrote these words on a strip of paper and laid the paper in the middle of the circle. Students then had to decide if their structures fit the description. The first student description was "a closed 3D structure." Students sorted their structures based on the property of closed vs. open. Most students had built a closed 3D structure; thus, the result of this first sorting activity was a large pile of almost all of students' structures. Mrs. B pointed out that the initial description applied to lots of their structures. She then invited a student with a more precise definition to read their description to the class. This students included pyramid in their description. Students then sorted their structures into a "pyramid" group and a "non-pyramid" group (Figure 2B). This sorting activity revealed that students had varied senses of what constitutes a pyramid. Some students included any structure with a pointy top or triangles; other students appealed to image prototypes, stating that their structure "looks like pyramid." Mrs. B and M.W. attempted to get children to share whether they agreed with how students had sorted their structure; however, the variability among students' structures and their perceptions of what constitutes a pyramid was so great that there was too little common ground to move forward.



(A) (B)

Figure 2: A) chart of student' descriptive words for 3D structures,
B) students sorting of their structures based on their sense of

Day 3-4: Comparing groups of structures and defining properties of shape

Because the variability of structures was overwhelming when including all students' structures the previous day, Mrs. B and M.W. chose examples of students' structures to represent two sets of structures. The first set of structures included examples of pyramids (hexagonal pyramid and square pyramid). The second set of structures included examples of prisms (a rectangular prism and a pentagonal prism). In both sets we elected to include an example of each class of shape that would likely be more familiar to students (a square pyramid and a rectangular prism) and a less familiar example (hexagonal pyramid and pentagonal prism) in order to expand students' image sets of each class of shapes. In the remainder of implementation summary we trace how the example sets of pyramids led to a classroom generated definition for pyramids; however, the development for prisms followed a similar pathway.

Mrs. B began the third day of instruction by showing students the two example pyramids and a chart with two columns—one for similarities and one for difference. Rather than telling students that both structures were types of pyramids, Mrs. B simply asked students to help describe what were similarities and differences between the structures. We decided based on students' sorting of their own structures the previous day that students did not yet understand names as classes of polyhedra; thus, we did not think it would be meaningful for students to compare the structures knowing they were types of pyramids. During this discussion, a number of opportunities for defining emerged as new properties were introduced and as disagreements about what properties the two examples shared. For example, some students said that that the square pyramid was closed and that the hexagonal pyramid was open. Previously, students described shapes as being open when you could "go inside the structure." Because the hexagon base was a frame with no cross bars and the square base had diagonal cross bars, students argued that you could fit your hand inside the base of the hexagonal pyramid but not the square pyramid. Other students disagreed, and argued that open meant that a structure was missing a face. To help resolve this disagreement Mrs. B brought back to open and closed tetrahedron from day 1, and asked students whether the open tetrahedron was "open" in the same way that the hexagonal prism. By comparing examples of two senses of open, students decided that a better way to define open was when a face was missing. Thus, this led to a revision in their chart. Instead of one structure being open and one structure being closed, they decided a new similarity was that both structures were closed.

A second defining opportunity arose when M.W. asked students what they meant by both structures being "pointy." While students pointed to the apex of each pyramids, others noted that the hexagonal pyramid was more "pointy." When asked what made one more "pointy," one students noted that the hexagonal pyramid only had one vertex. Other students quickly protested and began counting the number of vertices on each pyramid out loud. Mrs. B asked for a student to come up and point to the vertices they were counting to help others see what they thought was a vertex. By counting the number of vertices on each pyramid and ensuring that all students agreed on the number of vertices, students clarified what was meant by vertex beyond just corners with small angles to include all vertices regardless of angle measure. After establishing a new sense of vertices, M.W. turned back to the student that said that the hexagonal pyramid only had one vertex. She asked the student to point out what vertex she was referring to. When explaining why that vertex looks "different," the student explained that all of the triangles meet together and that's what makes it "pointy." M.W. took this opportunity to introduce the word "apex" to identify this as a special vertex where all the triangles meet. She then asked if the other structure also had an apex. Students all pointed out that the square pyramid also had an apex on the top vertex. To help further establish a definition for apex, Mrs. B rotated the square pyramid so that it was resting on a triangle face and the position of the apex was now longer appeared on the "top." Mrs. B then asked students which vertex was now the apex. Some students said that the pyramid now had two apexes—the two vertices closest to the ceiling, some students pointed to the top edge of the pyramid pointing towards the ceilings, and some students argued that the apex had not changed, "it turns with the structure." M.W. directed students' attention to the apex on the hexagonal pyramid, and asked students what made this the apex. Students recalled that they had noticed that all the triangles "met together." This helped students reach a consensus that a pyramid always has one apex where all triangles meet rather than just the top of the pyramid. These example of defining episodes show important development in students' understanding of both individual properties and relations between properties (i.e. number of faces and closure; vertices, apexes, and faces).

Day 4-6: Defining pyramids

Based on students' list of similarities and differences between the examples of pyramids, we hoped that students would next be able to generate a definition based on properties that can be generalized across all types of pyramids. We first thought it would be important to expand the example set of pyramids; thus, we asked students to construct examples of pyramids that we had not yet seen. From students' constructions we selected a number of examples and non-examples to present to the class. Mrs. B first showed students an octagonal pyramid (a pyramid with an octagon base), and she asked to students whether this new structure was also a pyramid like the previous two examples. We were surprised that some students argued that the structure was not a pyramid because it wasn't "pointy" or it couldn't have an octagon; however, other students argued that it was a pyramid because it still had an apex. Mrs. B told the class that their goal was to come up with a list of rules that would help them decide if the examples students had built were pyramids. She placed a new chart title, "rules of pyramids" next to their list of similarities and differences, and asked students what was a rule for pyramids. Students all agreed that the first rule was that all pyramids have an apex, where all the triangles come together. Mrs. B turned students' attention back to the octagonal pyramid and asked whether this could be a pyramid based on their first rule. Once students agreed that they could call this example a pyramid, Mrs. B pointed out that some people said that it couldn't have an octagon as a base. This led to the second rule, that a pyramid must have a base, but this base can be different shapes.

Next, Mrs. B introduced a non-example constructed by a student, a triangular prism. She asked students whether this new structure was also a pyramid. Students objected, appealing to their rules by stating that the structure does not a have an apex. M.W. suggested that maybe it could be a pyramid because it does have a base, however, students objected and said that a pyramid has to have an apex where all the triangles meet and the triangular prism didn't have an apex because it didn't just have triangles meeting together because it also had rectangle faces. This exchange led to the final rule, that all the faces (besides the base) need to be triangles.

After adding this rule, a student suggested that another rule is that pyramids cannot be "skinny." M.W. presented a "skinny" triangular pyramid made up of a small equilateral base and three-long isosceles faces to help demonstrate the students' claim. The student then explained that the example built by M.W. was not a pyramid because it was too skinny and there are no skinny pyramids in the real world, like in the desert. The rest of the class quickly objected, and argued that it was a pyramid because it fit all of their rules. Seeing that the student was upset by this interaction, M.W. took the opportunity to clarify a goal for defining in mathematics for students. She expressed that mathematicians often have very precise ways of talking about shapes that they help them build and imagine examples of shapes that maybe we don't see every day. This exchange helped solidify the position of students' definition as a tool within their classroom mathematics community.

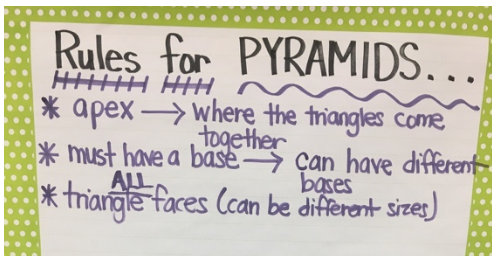


Figure 3: Classroom generated definition for pyramids

Post-interview results

At the conclusion of the instructional unit, M.W. conducted one-on-one post interviews with all 21 students. The interview tasks consisted of six 3D structure student needed to sort as pyramids, prisms, or neither (the structure is not a pyramid or a prism). Students also had to explain why they assigned a structure to particular group. M.W. made available to students a copy of the rules for prisms and pyramids that the class had generated as a tool to support students' reasoning and to analyze to what extent students understand the definitions as a tool.

Each student was shown six 3D structures (2 prisms, 2 pyramids, and 2 neither) listed in Table 1. The items were scored as correct or incorrect. Students reasoning were scored based on three categories: 1) systematic defining (student used definitions as a tool for explaining), 2) listing of properties (student identified properties but did appeal to definition), and 3) informal judgements (appealed to prototype reasoning).

90.5% of students were correct on greater than 80% of items (5 or more items), and 52.4% of student used definitions systematically on at least 80% of items. It is also notable that students that did not use definitions systematically on all items often improved in their use of definition throughout the interview. Given that this is a new practice for all students, these findings suggest that defining is accessible to young children, and we expect that in future iterations the number of students using definitions systematically will increase as we adjust the instructional unit to account for important forms of instructional support

Table 1: Overall Student Performance on Post Interviews

3D Structure	Correct	Use of Definition
Triangular pyramid	100%	86% (systematic) ;14% (lists properties)
Rhombic prism	62%	38% (systematic); 62% (lists properties)
Hexagonal anti-prism (neither)	86%	62% (systematic); 33% (lists properties); 5% (informal)
Cube (prism)	100%	71% (systematic); 29% (lists properties)
Octagonal pyramid	100%	86% (systematic); 14% (lists properties)
Rhombus base w/ triangle faces (neither)	86%	71% (systematic); 29% (lists properties)

Challenges

Despite greater attention to geometry in elementary mathematic standards, such as the Common Core, early elementary classrooms still devote little instructional time to this area of mathematics. This is in part due to issues of accountability testing that privilege number and arithmetic. Teachers and administrators feel greater pressure to devote the majority of instruction to these topics. However, while the goal is to develop a more integrated and cohesive mathematics instruction that privileges geometry, there is little research to help inform this transition towards a more spatialized mathematics curriculum.

One consequence of this tension between accountability and weak research base is a lack of school settings to conduct ongoing research. We have encountered numerous obstacles within schools to find partnering teachers who are interested in developing their geometry instruction. We also have faced obstacles at the district level from administrators who do not endorse and ultimately provide permission for research in geometry instruction because they see it as taking away from instructional time. Finally, as geometry continues to be positioned at the periphery of early elementary mathematics, preservice and teachers spend little professional development time improving their knowledge of geometry and understanding of children's thinking in relation to geometry. When we work with teachers, they are often learning the geometry content at the same time as their instructing. This creates challenges as it is difficult to support teachers in anticipating student thinking in order to provide instructional support when they have little sense of where instruction is leading. Thus, our design study supports understandings of way to support both children's and teachers' geometry learning.

Connection

Development within STEM fields increasingly rely upon new tools and methods of visualization; yet, current early elementary mathematics curriculum largely neglect cultivating children's mathematical visualization skills. Additionally, while the learning sciences have helped establish new ways of conceptualizing learning that emphasize students' access to disciplinary practice, we still understand little about how young children can learn to participate in these practices. Thus, development in both STEM fields and the learning sciences present new instructional goals and challenges regarding how to design new form of mathematical experiences for children. In our instructional design aimed at supporting young children's co-development of mathematical practices and concepts of shape and space, we attempted to address these new goals and challenges. Our design is greatly influenced by work in the learning science regarding what it means to engage in mathematics and the situated nature of visualization, but it is also focused on the horizon of where STEM fields are heading and the types of experiences that will prepare students for long term STEM learning.

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