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XIV. Notice on the connection between line geometry and the mechanics of rigid bodies

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Certain problems of the mechanics of rigid bodies are closely connected with the considerations of Plückerian line geometry, namely, the problem of the composition of arbitrary forces that act on a rigid body and the examination of the infinitely small motions that a rigid body can perform.

Plücker already spoke of such a connection in the first great communication that he made on his new geometric research [1]); soon afterwards, he devoted a special article to its elaboration [2]). Several places in his "Neuen Geometrie" [3]), in particular, numbers 25 and 39, related to the same situation.

As is obvious from repeated suggestions in "Neuen Geometrie," Plücker intended to expand upon the application of his geometric principles to mechanics in a larger, connected work that would follow "Neuen Geometrie." That Plücker did not realize this intention is even more regrettable than the aforementioned sparse locations where he expressed his mechanical concepts, although only very briefly and very hard to understand, as well as occasionally indeterminate. In the current, brief reference, I will now present the connection between the mechanical problem mentioned in the beginning statements with line geometry, in order to discuss some points that do not seem entirely clear to me in the Plückerian presentation. Along these lines, I will then discuss a special type of physical connection that can come about between force systems and infinitely small motions.

§ 1.

Some comments on the connection between the mechanics of rigid bodies and line geometry.

In order to see that an intimate connection between line geometry and the mechanics of rigid bodies in fact exists, one needs only to go over the geometric observations on the

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[1]) On a New Geometry of Space. Phil. Trans. 1865, pp. 725, cf., Additional Note. [Gesammelte Abhandlungen, Bd. 1].

[2]) Fundamental Views Regarding Mechanics. 1866. pp. 361. [Gesammelte Abhandlungen, Bd. 1]

[3]) Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement. Leipzig, B. G. Teubner. 1868/69.

mechanics of rigid bodies that were made by Poinsot, Möbius, Chasles, et al. [4]). In these investigations, one is incessantly involved with geometric things that enter into line geometry at a point that is both elementary and fundamental. A brief explanation might clarify this.

The matters in question relate in an essential way to the aforementioned two problems, the one treating the composition of forces that act on a rigid body, while the other examined the infinitely small motions that a rigid body performs. Both problems are, more or less, geometrically identical, insofar as the consideration of infinitely small motions of a rigid body comes down to the composition of infinitely small rotations *and infinitely small rotations compose by precisely the same rule as forces, namely, the parallelogram rule.* Thus, the concept of *force* appears to be coordinated [5]) with that of *infinitely small rotation*. All considerations that apply to forces that act on a rigid body can be applied in a completely analogous form to the infinitely small rotations that such a rigid body performs, and conversely.

Any given force that acts on a rigid body can always be replaced by *two* forces [6]), one of whom one can allow to act on an arbitrarily chosen line; from this, the line on which the other force acts and the intensities of both forces are then given. Likewise, any system of infinitely small rotations – or briefly, any infinitely small motion of a rigid body – can be composed of *two* infinitely small rotations. One of the rotational axes can therefore be chosen arbitrarily; the other axis and the magnitude of both rotations that are present are then determined. The lines of space are then pair-wise conjugate relative to a force system or an infinitely small motion. Möbius has shown [7]) that the opposite relation in both cases is given by a special type of dual relationship, of a type that v. Staudt referred to as a "null system," and is distinguished from other dual relationships by the fact that any point is united with the plane that corresponds to it. In a null system there is a triple infinitude of "null lines," which are those lines that coincide with their conjugates. All lines that run through an arbitrary point in its corresponding plane are null lines, and conversely, null lines are completely defined by the fact that they each lie in the corresponding planes to their points. If one would wish to allow one of the two forces into which a given force system can be decomposed to act along a null line of the null system in question then the second force would also act along the same null line. The intensity of both forces will be infinitely large and the forces seem to be oppositely directed. One is then dealing with a limiting case of the general decomposition of a force

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[4]) Cf., esp., Möbius: Lehrbuch der Statik. Leipzig 1837. Teil I. [Werke, Bd. III.] One finds a greater state of completeness in the relevant investigations of Schell's Theorie der Bewegung und der Kräfte. Leipzig 1870. [Cf., on this, in the name of recent literature, the article of Timerding in the Enzyklopädie der Math. Wiss., Bd. 4.]

[5]) This coordination of force and rotation is only mathematical, not physical. Cf. § 4.

[6]) The *force couples*, which played such an important role in Poinsot's theory, are thus to be regarded as (infinitely small) forces that act on an infinitely distant line. In fact, from Hebel's theorem, the two forces of the couple, when this expression is meaningful, have an infinitely distant (and thus, infinitely small) resultant. [Which was already shown by Culmann in the fourth edition of his Graphischen Statik (1864-1866).]

Analogously, in the considerations of the text the translations of a body are regarded as rotations about an infinitely distant axis.

Force couples and translations are then coordinated in the same sense as forces and rotations.

[7]) Crelles Journal, Bd. 10. Über eine besondere Art dualer Verhältnisse zwischen Figuren im Raume. [Werke, Bd. I.]

system into two forces that is meaningful only as a limiting case. By the consideration of infinitely small rotations one would likewise arrive at a limiting case that is not, in itself, meaningful when one lets it happen that one of the two rotations into which one can decompose the system is a null line of the associated null system. In both cases, the null lines also have yet another easily proven property. In the case of force systems, the null lines are those lines of space around which the moment of the force system is equal to zero; in the case of infinitely small motions, the null lines are those lines that will be perpendicular to themselves – i.e., without experiencing a displacement that is parallel to their direction [8]).

Now, the triply infinite manifold of null lines of a null system is precisely the same thing that Plücker referred to as a linear line complex [9]), i.e., a triply infinite manifold of lines whose coordinates satisfy a linear equation.

An arbitrary force system, like an infinitely small motion, is thus always linked with a well-defined linear complex. The lines of the complexes are those space lines, relative to which, the force system has no rotational moment (which are perpendicular to themselves for the infininetly small motion, resp.). The lines that are pair-wise associated by the complex – the *conjugate polars of the complex*, in the Plückerian terminology [10]) – are those line pairs on which two forces that are equivalent to the given force system can act (around which two infinitely small rotations can occur that replace the given infinitely small motion, resp.).

The arbitrary force system (the arbitrary infinitely small motion, resp.) can, for the sake of geometry, be replaced directly by the associated line complex. In the case of force systems, one must then consider only the absolute magnitude of the forces acting and regard all force systems as essentially identical that differ only with regard to their intensities. For infinitely small motions, the concept of infinitely small is correspondingly abstracted from the absolute values from the outset, and therefore such an abstraction does not need to be expressly introduced here.

A linear complex can, in particular, become a special complex [11]) – i.e., to go over to the totality of all lines that intersect a fixed line. The force system that is equivalent to the linear complex correspondingly reduces to a single force that acts on the fixed lines; the infinitely small motion that is equivalent to the linear complex corresponds to a rotation that comes about around the fixed line.

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[8]) Möbius, Statik. I. § 84, et seq. [Werke, Bd. 3] Chasles, in the Comptes rendus. 1843. Sur les mouvements infinitement petits des corps.

[9]) Cf., Plücker. Neue Geometrie, no. 29.

[10]) Neue Geometrie, no. 28.

[11]) Neue Geometrie, no. 45. This corresponds to the case where the determinant of the null system in question vanishes.

§ 2.

Analytical representation. Coordinates of forces and (infinitely small) rotations. Coordinates of force systems and (infinitely small) motions.

One proves the statements we previously discussed most simply when one, like Plücker [12]), endows a force (infinitely small rotation, resp.) with virtual *coordinates*, namely, those coordinates possessed by the straight line along which the force acts (around which the rotation occurs).

On the basis of a rectangular coordinate system, if we let x, y, z and x', y', z' be the coordinates of two points of the line thus determined then, according to Plücker, their connecting line takes on the coordinates given by the relative values of the following six expressions:

$$\rho X = x - x', \qquad \rho Y = y - y', \qquad \rho Z = z - z',$$
$$\rho L = yz' - y'z, \qquad \rho M = x'z - xz', \qquad \rho N = xy' - x'y.$$

Thus, one has, identically:

$$XL + YM + ZN = 0,$$

from which, the relative values of the six quantities X, Y, Z, L, M, N that determine a line necessarily come down to four quantities.

It is known that one comes to *the same* six coordinates when one regards the line, not, as we have, as the connecting line between two points – a *ray*, as Plücker says – but as the intersection line of two planes – an *axis*, in Plücker's terminology. While the mechanical concept of force acting along a line is connected with the geometric one of the line as a ray, the mechanical notion of a rotation occurring around a line is connected with the geometric one of the line as an axis.

We now endow a force that acts along a line (an infinitely small rotation that happens around it, resp.) with the six virtual quantities:

$$X, Y, Z, L, M, N$$

as coordinates.

For forces, this way of determining coordinates comes about in the usual way, as long as one still establishes that the absolute values of the six coordinates are proportional to the intensities of the force, as one does for force in mechanics. X, Y, Z are the components parallel to the three coordinate axes, while L, M, N are the rotational moments around the same axes.

Such a way of establishing the absolute values for the coordinates has no meaning for an infinitely small rotation as long as one agrees that no other (arbitrarily chosen) infinitely small rotation that one gives the intensity 1 is equivalent to it.

[12]) Cf., Neue Geometrie, no. 25.

One now composes forces and rotations whose coordinates have been obtained in this way as absolute values by adding their coordinates [13])

This theorem first expresses the fact that all systems of forces (all systems of rotations, resp.) that yield the same six values upon addition of the coordinates of the individual forces (rotations, resp.) are equivalent.

These six values that we obtain by addition, which we can speak of as the *coordinates of the force system* (*the coordinates of the infinitely small motion*, resp.), may be called:

$$\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}.$$

In particular, one can have:

$$\Xi\Lambda + \mathrm{HM} + \mathrm{ZN} = 0.$$

The force system can then be replaced with a single force and the infinitely small motion with a rotation whose virtual coordinates are:

$$X = \Xi, \qquad Y = \mathrm{H}, \qquad Z = \mathrm{Z}, \qquad L = \Lambda, \qquad M = \mathrm{M}, \qquad N = \mathrm{N}.$$

However, if the condition:

$$\Xi\Lambda + \mathrm{HM} + \mathrm{ZN} = 0$$

is not fulfilled then one can only replace the force system with two forces and the infinitely small motion with two rotations. If the coordinates of the two forces (the two rotations, resp.) are:

$$X', Y', Z', L', M', N' \quad \text{and} \quad X'', Y'', Z'', L'', M'', N'',$$

then one must have:

$$\begin{aligned} X' + X'' &= \Xi, & Y' + Y'' &= \mathrm{H}, & Z' + Z'' &= \mathrm{Z}, \\ L' + L'' &= \Lambda, & M' + M'' &= \mathrm{M}, & N' + N'' &= \mathrm{N}. \end{aligned}$$

One likewise has:

$$X' L' + Y' M' + Z' N' = 0, \qquad X'' L'' + Y'' M'' + Z'' N'' = 0.$$

These equations now express the fact that the straight lines X', Y', Z', L', M', N' and $X'', Y'', Z'', L'', M'', N''$ shall be conjugate polars relative to the linear complex whose equation is:

$$\Lambda X + \mathrm{M}Y + \mathrm{N}Z + \Xi L + \mathrm{H}M + \mathrm{Z}N = 0.$$

[13]) Battaglini has carried out the arguments connected with the present matters in a series of articles in the Rendiconti and Atti of the Academy of Naples on the basis of tetrahedral coordinates. (In the Rendiconti of February, May, August 1869, May 1870, and the Atti, v. 4, 1869.)

One also confers Cayley: On the six Coordinates of a line. Cambridge Transactions, v. 8, 1868, esp., chapter: Statical and Kinematical Applications, pp. 25. [Coll. Papers, v. VII.]

Since one does not find a vantage point in the "New Geometry" from which one see this with no further explanation, a brief proof follows here.

The condition that a straight line intersects the fixed line X', Y', Z', L', M', N' is:

$$L'X + M'Y + N'Z + X'L + Y'M + Z'N = 0.$$

Likewise for the line $X'', Y'', Z'', L'', M'', N''$ one has:

$$L''X + M''Y + N''Z + X''L + Y''M + Z''N = 0.$$

This says: All lines that intersect both X', Y', Z', L', M', N' and $X'', Y'', Z'', L'', M'', N''$ belong to the linear complex in question. However, this is the characteristic property [14]) of the conjugate polars. Q.E.D.

One can virtually refer to the coefficients:

$$\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$$

in the equation for the linear complex as the *coordinates of linear complexes* [15]). This is permissible because when the six coefficients satisfy the equation:

$$\Xi\Lambda + \mathrm{HM} + \mathrm{ZN} = 0,$$

they are the coordinates of that special complex that is represented by the given linear equation:

$$\Lambda X + \mathrm{M}Y + \mathrm{N}Z + \Xi L + \mathrm{H}M + \mathrm{Z}N = 0;$$

i.e., the coordinates of all straight lines that are intersected by all of the lines that satisfy the equation. The present equation must, in fact, be verified in order for two lines with the coordinates $\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$ and X, Y, Z, L, M, N to intersect.

We now use this expression: If we coordinate a linear complex, regardless of whether the complex is or is not a special one (a straight line), then we can, independently of whether a given force system does or does not have a resultant (whether a given infinitely small motion does or does not reduce to a rotation, resp.), state the theorem:

The geometric structure for a force system (an infinitely small motion, resp.) with the coordinates:

$$\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$$

is a linear complex that possesses the same coordinates.

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[14]) Neue Geometrie, no.28.

[15]) Cf.: "Die allgemeine lineare Transformationen der Linienkoordinaten." Math. Annalen, Bd. 2. [See Abh. III of this collection.]

§ 3.

Discussion of some particular issues.

As we already mentioned in the Introduction, there are some aspects to the presentation that Plücker gave of the connection between his line geometry and the mechanics of rigid bodies that did not seem to be clear, as one finds in perhaps no. 25 of "Neuen Geometrie."

From the foregoing, force systems and infinitely small motions can both be described by the same geometric structure of the linear complex. One and the same complex is therefore, on the one hand, a picture of a force system, and, on the other, a picture of an infinitely small motion: This is completely similar to the way that one and the same straight line (a special complex) can have, on the one hand, a force acting on it and, on the other, a rotation taking place around it. Between the force system and the infinitely small motion, which relate to the same linear complex, there exists little to suggest a causal physical connection between the forces that act on a line and the rotations that take place around it.

However, in the Plücker presentation it appears that the force system is the cause of the associated (relative to the same linear complex) infinitely small motion. Correspondingly, both of them will be essentially identical and are given the common name "Dyname."

The fact that a force system and an infinitely small motion cannot be causally connected at all by these considerations is obvious from the fact that a well-defined motion of a rigid body results from a given force system only when the body has a mass, a center of mass, an inertia ellipsoid, etc. By means of a well-defined rigid body whose mass, etc. are given once and for all, each force system will then be generally coordinated with a well-defined infinitely small motion as its effect [16]). However, as long as one speaks only of a rigid body whose mass distribution does not come into consideration, one can not, by any means, speak of a causal association.

There thus arises the question of whether another sort of physical connection between force systems and infinitely small motions prevails that would lead back to the mathematical coordination of the two things. This shall be treated in the next paragraphs.

A further point that does not seem entirely clear in the Plücker presentation is the following one, which is usually closely connected with the one mentioned:

When we speak of a force, this is linked with the geometric notion of a line as a series of points – i.e., as a ray. On the other hand, when an (infinitely small) rotation takes place around a line we regard the line as a pencil of planes – i.e., as an axis. To link an axis with the concept of a force – i.e., to therefore think of a force that would rotate a rigid body around a *well-defined* line – is just as impossible as linking a ray with the concept of an infinitely small motion that must displace the body along a single, *well-defined* line. A torque – i.e., a force-couple – does not have one axis, but infinitely many

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[16]) Since the force system, as well as the infinitely small motion, can be replaced with a linear complex, any rigid body with a given mass distribution is the basis for a special type of spatial relationship, by which any linear complex is associated with a second one. For bodies whose inertia ellipsoid is a sphere, the relationship will be essentially that of polar reciprocity with respect to a sphere that is centered on the center of mass.

of them whose direction alone is given; likewise, a translation does not displace a single ray into itself, but displaces all parallel rays simultaneously. Thus, one cannot link the geometric concept of an axis with the mechanical one of a torque either, or the geometric concept of a ray with the mechanical one of a translation. One can speak only of forces that act *along* straight lines and motions that occur *around* them. This does not emerge clearly in Plücker; Plücker spoke mostly of forces and rotations, and then again also of translations and torques, and gave the impression in various places that any force was causally linked with a translation and any torque, with a rotation.

That one can speak only of forces that act *along* straight lines, and motions that occur *around* them goes back to the non-dualistic character that the metric does indeed possess. As is well-known, the same thing is true for a certain plane curve of second degree, namely, the infinitely distant imaginary circle. However, one can now conceive of a general metric from the process given by Cayley [17]), by which a general surface of second degree plays the same role as that of the aforementioned curve. If one introduces such a metric and simultaneously replaces the six-fold infinitude of motions of our space with just as many linear transformations that leave the fundamental surface of second degree unchanged [18]). – One can equally speak of forces that act along lines and around them, and of motions that take place around lines and along them. However, both types of forces and motions would then be equivalent. A rotation around a line is then the same thing as a displacement along the conjugate polar to it relative to the fundamental surface of second degree. Likewise, a force that acts along a line is the same thing as a force that endeavors to rotate around its conjugate polar.

A third point that is not explicitly brought up by Plücker is the fact that one may bring only infinitely small rotations, and certainly infinitely small motions under consideration. Finite motions certainly compose in a different manner than the infinitely small ones, so – for example – the order of operations in the composition is not irrelevant, while this sequence is not an issue with infinitely small motions, and likewise for the composition of forces.

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[17]) Cf., Cayley. A sixth Memoir upon Quantics. Phil. Trans. 1859. [Coll. Papers, Bd. II.] Also, Salmon's Analytische Geometrie der Kegelschnitte. Chap. XXII (the Fiedler translation).

[18]) A surface of second degree goes to itself under a six-fold infinitude of linear transformations. However, they split it into two six-fold infinite manifolds. *The two systems of rectilinear generators of the surface remain unchanged under the transformations of the one manifold, while the transformations of the second manifold switch the two systems oppositely.* The transformations of the former manifold are intended in the text; if the F_2 gradually goes over to the infinitely distant imaginary circle then they gradually go over to the six-fold infinitude of motions of space. – On a later occasion, I think I will show how the Cayley metric mentioned in the text, with the addition of this six-fold infinitude of transformations, leads to precisely the same geometric notions that Lobatchewsky and Bolyai have developed from a completely different starting point.

§ 4.

On the type of physical connection between force systems and infinitely small motions.

Now, there is, in fact, a physical connection between force systems and infinitely small motions that explains in what way the two things come to be mathematically coordinated. This relation is not of a sort that it associates each force system with a single infinitely small motion, but of another sort that is *dualistic*.

Let there be given a force system with the coordinates:

$$\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$$

and an infinitely small motion with the coordinates:

$$\Xi', \mathrm{H}', \mathrm{Z}', \Lambda', \mathrm{M}', \mathrm{N}',$$

where one might have determined the coordinates absolutely in the manner described in § 2. *Then*, as we shall not further elaborate upon here, *the expression:*

$$\Lambda' \, \Xi + \mathrm{M}' \, \mathrm{H} + \mathrm{N}' \, \mathrm{Z} + \Xi' \, \Lambda + \mathrm{H}' \, \mathrm{M} + \mathrm{Z}' \, \mathrm{N}$$

represents the quantum of work that the given force system performs during the given infinitely small motion [19]).

In particular, if:

$$\Lambda' \, \Xi + \mathrm{M}' \, \mathrm{H} + \mathrm{N}' \, \mathrm{Z} + \Xi' \, \Lambda + \mathrm{H}' \, \mathrm{M} + \mathrm{Z}' \, \mathrm{N} = 0$$

then the given force system performs *no* work during the infinitely small motion.

When we, on the one hand, consider the $\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$, *and, on the other hand, the* $\Xi', \mathrm{H}', \mathrm{Z}', \Lambda', \mathrm{M}', \mathrm{N}'$*, to be variable this equation now represents the connection between force systems and infinitely small motions.*

If we consider the $\Xi, \mathrm{H}, \mathrm{Z}, \Lambda, \mathrm{M}, \mathrm{N}$ to be variable then the equation says: There is a four-fold infinitude of force systems [20]) that perform no work under a given infinitely small motion. Their coordinates must satisfy a homogeneous linear equation. One can also express this by saying:

An infinitely small motion will be represented by a homogeneous, linear equation between the coordinates of a force system; in completely the same sense, as perhaps in

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[19]) [The analogy between configurations and infinitely small motions was already found by Rodrigues in his great treatise: Les lois géometriques qui régissent le deplacement d'un système solide, etc. (Liouvilles Journal, t. 5, 1840), where it led back to the principle of virtual velocities along a general train of thought. Cayley then found (in the aforementioned essay on the six coordinates of a general line) analytical formulas that are similar to the ones here in the text. All that was missing was the discussion of the geometry of linear complexes.

See, moreover, as following through with the argument suggests, the essay XXX that follows below.]

[20]) They are considered to be identical when they differ only with respect to the absolute values of their coordinates.

analytical geometry, a certain plane will be represented through a homogeneous linear equation in the point coordinates that expresses the fact that the distance between a point and that certain plane should be equal to zero.

On the other hand, if we consider $\Xi', H', Z', \Lambda', M', N'$ to be variable then our equation expresses that: There is a four-fold infinitude of infinitely small motions during which a given force system performs no work. Their coordinates satisfy a homogeneous, linear equation. In other words:

A force system will be represented by a homogeneous linear equation between the coordinates of an infinitely small motion, which completely corresponds to the way that, when we remain in the previous example from analytical geometry, a point will be represented by a linear equation between plane coordinates.

The geometric substrate for the *duality between force systems and infinitely small motions* that is set forth herewith defines the double type and manner by which one may regard them as concepts in line geometry when one starts with the linear complex as a spatial element [21]). A linear complex is then, at the same time, a spatial element and a structure represented by a linear equation. The equation:

$$\Lambda' \,\Xi + M' \,H + N' \,Z + \Xi' \,\Lambda + H' \,M + Z' \,N = 0,$$

in which, moreover, Ξ, H, Z, Λ, M, N and Ξ', H', Z', Λ', M', N' shall mean the coordinates of two linear complexes, expresses a relation that is true between them, which I have referred to as an *involutory* relationship [22]). What one must understand by the involutory relationship between two linear complexes is perhaps simplest to explain. Each point of space corresponds to a plane in the first complex, while this plane corresponds to a point in the second complex. One comes to the same point by the involutory relationship of the two complexes when one considers the plane that corresponds to the initially chosen points in the second complex and now seeks the point that this plane is associated with in the first complex. The dual conversions of space that are linked with the two complexes are thus mutually interchangeable. – If one of the two complexes is special – i.e., a line – then the involutory relationship enters in when the line belongs to the other complex. If both complexes are lines then the involutory relationship is equivalent to the intersection of the two lines.

A linear complex can, from what we said, be regarded as either a spatial element or as the four-fold infinite manifold of the complex that is in involution with it.

If we give the linear complex the mechanical meaning of an infinitely small motion then the four-fold infinitude of the complex that is in involution with it represents the force system that performs no work under the infinitely small motion. Conversely: We link the linear complex with the mechanical notion of a force system then the four-fold infinitude that is in involution with it represent the infinitely small motions for which the force system does no work.

[21]) I have carried out these geometric considerations further in the previously-cited essay: "Die allgemeine lineare Transformationen der Linienkoordinaten." Math. Ann., Bd. 2. [See Abh. III of this collection.]

[22]) Battaglini also arrived at the involutory relationship between two linear complexes in the cited essays; he referred to it as a "harmonic relationship" between two complexes. The word "involution" was used by him in order to refer to manifolds that depend linearly upon parameters.

Within the four-fold infinitude of a complex that is in involution with a linear complex, as we already said, one also finds the three-fold infinitude of straight lines that are associated with it. We can especially express the previous two theorems for it. They will then be equivalent with the two theorems stated in § 1, which read: When a linear complex is the image of a force system or an infinitely small motion then the lines that are associated with it are the lines of space relative to which the force system has no rotational moment (the infinitely small motion will be perpendicular to itself, resp.).

Göttingen, in June 1871.

[Essay XIV has been placed at the end of the part on line geometry because in § 3 it includes my first communication on the meaning of the Cayley metric for non-Euclidian geometry and thus defines a natural transition from the investigations to the foundations of geometry. The question that I posed in § 3 was soon thoroughly treated by Lindemann, who went to Erlangen with me in Fall 1872, in his dissertation (Über unendlich-kleine Bewegungen und über Kraftsysteme bei allgemeiner projektivescher Massbestimmung, Math. Ann. Bd. 7, 1873). K]